Figure 11: a The traditional setup. The vertical light curtain.

Figure 12: b The new light curtain. — lower background light intensity; — increased effective width of curtain; — same power / same light source.

prob8.tex **Problem 8**

The Rolling Carpet
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Let us consider an ideal carpet with the following data:

d — thickness of the carpet,

l — length of the carpet,

x — length of the unrolled part of the carpet at any time t,

R — radius at $t = 0$,

r — radius at $t > 0$,

M — mass at $t = 0$,

m — mass at $t > 0$,

ω_0 — angular velocity at $t = 0$,

ω — angular velocity at $t > 0$,

$\Theta_0 = \frac{1}{2}MR^2$ — moment of inertia at $t = 0$,

$\Theta = \frac{1}{2}mr^2$ — moment of inertia at $t > 0$.

Figure 1:

Two geometrical conditions hold for the rolling carpet:

1. The sum of the area of the whole carpet is constant at any time:
$$r^2\pi = R^2\pi - xd\ .$$

From this equation we can obtain r as a function of x:

$$r^2(x) = R^2(1 - x/l) \ .$$

2. There is no relative motion between the floor and the part of the carpet which contacts it:

$$v = \omega r \ ,$$

(v is the velocity of the center of mass of the rolling part of the carpet).

The carpet possesses different forms of energy during the motion:

$$E = \frac{1}{2}mv^2 + \frac{1}{2}\Theta\omega^2 + E_{\mathrm{P}} + E_{\mathrm{L}}$$

($\frac{1}{2}mv^2$ — translational energy, $\frac{1}{2}\Theta\omega^2$ — rotational energy, E_{P} — potential energy, E_{L} — energy loss).

From this equation we can calculate the energy of the carpet at any time t:

$$E = \frac{1}{2}mv^2 + \frac{1}{2} \cdot \frac{1}{2}mr^2 \cdot \frac{v^2}{r^2} + E_{\mathrm{P}} = \frac{3}{4}mv^2 + E_{\mathrm{P}} \ .$$

The energy of the carpet at $t = 0$:

$$E_0 = \frac{3}{4}mv_0^2 + E_{\mathrm{P}_0}$$

(E_{P_0} is the potential energy of the carpet at $t = 0$).

I. In this part we assume that there is no energy loss during the motion and we neglect the change of the potential energy as well. Because of the conservation of energy, the following equation holds at any time:

$$E_0 = \frac{3}{4}Mv_0^2 = \frac{3}{4}mv^2 \ .$$

Because of the homogeneity of the carpet $m = Mr^2/R^2$ holds, from which we obtain:

$$\frac{3}{4}Mv_0^2 = \frac{3}{4}M\frac{r^2}{R^2}v^2 \ .$$

From this equation we obtain the velocity v as a function of x:

$$v = \frac{v_0}{\sqrt{1-x/l}} \ .$$

In Fig. 1 we can see v as a function of x. In this case the carpet unrolls completely because v never reaches zero during the motion.

II. In this part we still neglect the potential energy of the carpet but we assume that there is an energy loss because of the deformation of the carpet during rolling out. We also assume that this energy is proportional to $1/r$ and to the length of piece of the carpet. The coefficient of this proportionality k depends on the material the carpet is made off and on d. We obtain the energy loss as a function of x by integrating $1/x$ from 0 to x:

$$E_{\mathrm{L}} = k\int_0^x \frac{1}{r}\mathrm{d}x = k\int_0^x \frac{1}{R\sqrt{1-x/l}}\mathrm{d}x$$

$$= \frac{k}{R}\left[\sqrt{1-x/l}\right]_0^x \left(\frac{-2R^2\pi}{d}\right) = 2\frac{k}{R}l\left(1-\sqrt{1-x/l}\right) \ .$$

The energy loss during the whole motion ($x = l$ and $dl = R^2\pi$):

$$E_{\mathrm{L}}' = \frac{2kR\pi}{d} \ .$$

Because of the conservation of energy, the carpet will unroll completely if $E_0 \geq E_{\mathrm{L}}'$:

$$\frac{3}{4}Mv_0^2 \geq \frac{2kR\pi}{d} \ .$$

From this we obtain the condition for the initial velocity v_0. If

$$v_0 \geq \sqrt{\frac{8kR\pi}{3dM}} \ ,$$

the carpet possesses more energy than the energy loss and consequently it will unroll completely. If the velocity at $t = 0$ is smaller than this limit value, the carpet will stop before it has unrolled completely. We can see this in Fig. 2, where the relative velocity v/v_0^* is shown as a function of x $\left(v_0^* = \sqrt{8kR\pi/3dM}\right)$.

III. In this case we also take into account the potential energy of the carpet. The initial magnitude of it is

$$E_{\mathrm{P}_0} = MgR \ .$$

At time t we can write the potential energy as a sum of the potential energy of the rolling part of the carpet and that of the unrolled part:

$$E_{\mathrm{P}} = mgr + (M - m)g(d/2) \ .$$

Using $r = R\sqrt{1 - x/l}$ and $m = Mr^2/R^2$ we obtain the following equation:

$$E_{\mathrm{P}} = MgR\left(1 - \frac{x}{l}\right)^{3/2} + Mg\frac{d}{2}\frac{x}{l} \ .$$

Due to the conservation of energy

$$E_{\mathrm{P}_0} + E_0 = E_{\mathrm{trans}} + E_{\mathrm{rot}} + E_{\mathrm{P}} + E_{\mathrm{L}} \ ,$$

where E_{trans} — translational energy, E_{rot} — rotational energy. Substituting the energies obtained before we obtain:

$$MgR+\frac{3}{4}Mv_0^2 = \frac{3}{4}M\frac{r^2}{R^2}v^2+Mgr\left(1 - \frac{x}{l}\right)^{3/2}+Mg\frac{d}{2}\frac{x}{l}+2\frac{k}{R}l\left(1 - \sqrt{1 - \frac{x}{l}}\right) \ .$$

From here we obtain v as a function of x:

$$v^2 = \frac{v_0^2 - 43Rg(1 - x/l)^{3/2} - 43gd2xl - 83klRM\left(1 - \sqrt{1 - x/l}\right) + 43gR}{1 - x/l} \ .$$

In Figs. 3–6 we can see the velocity v as a function of x for different ratios of the initial potential energy of the carpet E_{P_0} and the energy loss E'_L during the motion. For too small initial velocities the carpet stops before it has unrolled completely. If the potential energy is large compared to the energy loss, the carpet unrolls for any initial velocity v_0 (Figs. 5 and 6).