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Cathode-ray tube

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10 The time dependence of pressure

First of all we introduce two quantities which are appropriate for description of a gas flow through a duct (a tube).

(a) *The flow of gas* Q as a gas-flow rate through the cross-section of a duct in a time unit:

$$Q = \frac{G}{t} = \frac{pV}{t} \qquad \left(\text{exactly} \quad p\frac{\mathrm{d}V}{\mathrm{d}t}\right) \tag{16}$$

(t — time, p — gas pressure, V — gas volume, $G = pV$).

(b) *Vacuum conductance* C as a quantity which characterizes a ratio of the flow of gas Q to the pressure difference which causes the flow:

$$C = \frac{Q}{p_1 - p_2}\ . \tag{17}$$

It means that the vacuum system with the conductance C, at one end of which the pressure is zero, at its other end reveals the pumping velocity (in $m^3 s^{-1}$) which is equal to the conductance of the system. This quantity has the advantage (in opposite to the flow of gas) that it does not depend on pressure diffrence between the ends.

It is necessary to examine carefully the course of the flow of air through the hole into the cathode-ray tube to describe the pressure changes in time. We can simulate this hole almost perfectly by a cylinder with the length l and the radius r, because the laser is the only one method for shaping of this hole and the laser makes a cylindrical hole. Thus we will call it "the tube".

10.1 Viscous flow

The gas behaves like a continuous medium at pressures at which the mean-free-path of molecule λ is lower than the minimal dimension of the system, what means

$$\lambda < 2r. \tag{18}$$

During the flow (it means during the movement of the gas through the tube), there are layers of gas, where gas moves with the same velocity u. They have shape of cylindrical thin-walled tubes, because the flow of gas is cylindrically symetrical in the cylindrical tube. Because of the frictional force, the gas layer which adjoins the tube wall has the lowest velocity, and we will assume that it equals zero. The highest velocity of gas is obviously at the tube axis. There exists interaction between adjenct layers with different velocity — internal friction of the gas (viscosity). We call this kind of flow *viscous flow.*

The force f acting on the layer with the radius x and the thickness dx is

$$f = 2\pi x \ \mathrm{d}x(p_1 - p_2) \ , \tag{19}$$

where p_1 and p_2 are pressures at the ends of the tube. If we want to determine the velocity of this layer, we have to determine frictional forces between our layer and adjacent (internal and external) layers. Frictional force F, acting on the surface unit of the layer, is proportional to the velocity gradient, which is perpendicular to the surface of the layer, and to viscosity coefficient η:

$$F = -\eta \frac{\mathrm{d}u}{\mathrm{d}x} \ .$$

Frictional force F_1 acting on the internal surface of the layer is determined by the product of F and the surface area $2\pi xl$:

$$F_1 = -2\pi x l \eta \frac{\mathrm{d}u}{\mathrm{d}x}. \tag{20}$$

A negative sign means that the velocity decreases for increasing x (in the axis of the tube $x = 0$ and there is maximum velocity). This force accelerates the layer and brakes the internal adjacent layer, which moves faster. Frictional force F_2, acting on the external surface of the layer, is:

$$F_2 = -2\pi l \eta (x + \mathrm{d}x) \cdot \left(\frac{\mathrm{d}u}{\mathrm{d}x} + \frac{\mathrm{d}^2 u}{\mathrm{d}x^2} \mathrm{d}x \right). \tag{21}$$

This force brakes the layer and accelerates the external adjacent layer, which moves slower. Forces f, F_1 and F_2 have to be in equilibrium during the stationary flow, which does not change in time:

$$f + F_1 - F_2 = 0 . \tag{22}$$

By the substitution of (5), (6) and (7) we obtain

$$2\pi \left[x \mathrm{d}x (p_1 - p_2) - x l \eta \frac{\mathrm{d}u}{\mathrm{d}x} + l\eta (x + \mathrm{d}x) \cdot \left(\frac{\mathrm{d}u}{\mathrm{d}x} + \frac{\mathrm{d}^2 u}{\mathrm{d}x^2} \mathrm{d}x \right) \right] = 0.$$

If we neglect the therm with $\mathrm{d}x^2$, we get

$$\eta l \frac{\mathrm{d}u}{\mathrm{d}x} + \eta l x \frac{\mathrm{d}^2 u}{\mathrm{d}x^2} = \eta l \frac{\mathrm{d}}{\mathrm{d}x} \left(x \frac{\mathrm{d}u}{\mathrm{d}x} \right) = -x(p_1 - p_2).$$

By the first integration we obtain

$$x \frac{\mathrm{d}u}{\mathrm{d}x} = -\frac{x^2}{2\eta l}(p_1 - p_2) + C_1 .$$

We substitute integrational constant $C_1 = 0$, because the velocity is maximal at the axis of the tube (it means $\mathrm{d}u/\mathrm{d}x = 0$) and we get

$$\frac{\mathrm{d}u}{\mathrm{d}x} = -\frac{x(p_1 - p_2)}{2\eta l},$$

Figure 2: Velocity u of gas layers flowing through a tube versus the distance from the tube axis.

By the second integration we obtain

$$u \;= -\frac{x^2(p_1 - p_2)}{4\eta l} + C_2.$$

The velocity is zero at the wall of the tube, thus it must be

$$C_2 = \frac{r^2(p_1 - p_2)}{4\eta l}\ .$$

We get the final expression for u

$$u \;= \frac{p_1 - p_2}{4\eta l}(r^2 - x^2), \tag{23}$$

which determines the dependence of the velocity u on the distance of the layer x from the center line (axis): this dependence has a parabolic shape (Fig. 1).

We calculate the volume of gas V', which passes through the tube in the unit of time, by the integration of product of the velocity and the area of a corresponding element of the cross-section:

$$V' = \int_0^r u\ 2\pi x\ \mathrm{d}x = \int_0^r \frac{p_1 - p_2}{4\eta l}(r^2 - x^2)2\pi x\ \mathrm{d}x = \frac{\pi r^4}{8\eta l}(p_1 - p_2). \tag{24}$$

This is so called *Hagen-Poiseuille's law*. We obtain the expression of the flow Q by the multiplication of V' by the mean value of pressure $p = (p_1 + p_2)/2$:

$$Q = pV' = \frac{\pi r^4}{16\eta l}(p_1^2 - p_2^2). \tag{25}$$

This relation involves the dependence of the flow Q on a kind of gas (viscosity coefficient). The flow of gas increases with the fourth power of a radius of the tube and increases linearly with the pressure p. In our case for the diameter of the hole 1 μm and the thickness of glass, which in produced color cathode-ray tubes is $8 - 15mm$, this model is fully sufficient.

10.2 Molecular flow

If the mean-free-path of molecules is higher than the hole diameter it is not possible to use the previous description of the flow. In these conditions, there are almost no collisions and this is not possible to consider gas as a continuous medium and we can not talk about single layers, which interact by frictional forces. At very low pressures molecules, which move through the tube, change their linear momentum only during collisions with the tube wall. Gas flow depends on interactions of molecules with the wall and we can talk about *external friction of gas.* This kind of flow is called *molecular flow.*

The force f, which causes the molecular flow, equals

$$f = \pi r^2 (p_1 - p_2).$$

This force is (in a stationary flow) in equilibrium with the intensity of momentum transfered from molecules to the wall per 1s:

$$F = Anmu,$$

where A is the total area of the internal tube wall, n is a number of molecules, that impinge on the unit of the wall surface per 1s, m is the mass of a molecule, u is the velocity of flow. Substituting

$$n = \frac{1}{4} \nu v_a$$

(ν is concentration of molecules, v_a is mean arithmetic velocity) and $A = 2\pi rl$, we obtain

$$F = \frac{1}{2} \pi rlm\nu v_a u \ .$$

In the stationary state should be

$$f = F \ ,$$

what means

$$\pi r^2 (p_1 - p_2) = \frac{1}{2} \pi r l m \nu v_a u.$$

If we solve this expression, we obtain

$$u = \frac{2r(p_1 - p_2)}{l m \nu v_a} \ . \tag{26}$$

After substitution of an expression for ν from $p = \nu k T$ (k is the Boltzmann constant and T is the absolute temperature) and

$$v_a = \sqrt{\frac{8kT}{\pi m}} \tag{27}$$

we obtain

$$u = \frac{r(p_1 - p_2)}{lp} \sqrt{\frac{\pi k T}{2m}}, \tag{28}$$

where $p = \frac{1}{2}(p_1 + p_2)$ is the mean value of pressure. The volume of gas V', that passes through the tube per 1s is given by the product of the cross-section area and the velocity u:

$$V' = \pi r^2 u = \frac{\pi r^3}{l} \frac{p_1 - p_2}{p} \sqrt{\frac{\pi k T}{2m}}.$$

We used the expression (13) for v_a, but in fact it is valid only for the equilibrium state. The assumption concerning the mean value of momentum is not satisfied too, because gas is not at rest. If we assume that molecules do ordered movement and their velocity is u, we can obtain a result which differs by an empirical factor (instead of π we use $\frac{8}{3}$):

$$V' = \frac{8}{3} \frac{p_1 - p_2}{p} \frac{r^3}{l} \sqrt{\frac{\pi k T}{2m}} \ . \tag{29}$$

The flow according to (2) is in this case

$$Q = \frac{8}{3}(p_1 - p_2)\frac{r^3}{l}\sqrt{\frac{\pi kT}{2m}} \ . \qquad (30)$$

In the case of molecular flow the flow is proportional to the cube of the tube radius in contrast to the relation for viscous flow (11), where the flow is proportional to the fourth power of the radius. It causes that less gas flows through the same system at low pressures (according to (16)) than at higher pressures (11). In the case of the molecular flow the gas flow is directly proportional to the square root of absolute temperature and inversely proportional to the square root of the molecular mass. The flow is independent of a viscosity coefficient and of the mean value of pressure.

Similarly as in the case of Hagen-Poiseuille's law, Eqn.(16) is not valid for holes where the length is almost the same as the radius r.

Now we can apply these results to the flow in the tube. At first we have to decide, if the flow is molecular or viscous. It is not possible to answer this question unambiguously, because the pressure of gas changes along the tube in a wide range (from the athmospheric pressure to very low pressure inside the cathode-ray tube). Air flow through one part of the tube is viscous and through another part is molecular. The extent of both parts depends on pressures at the ends of the tube (p_0 — initial pressure in the cathode-ray tube, p_l — atmospheric pressure), on the length l and on the cross-section area πr^2 (and also on a kind of gas, if no air leaks in the system) — Fig. 2.

Figure 3: Scheme of the tube-like hole.

We define p_x as a pressure, at which the mean-free-path of molecule

λ equals to the diameter of the tube $2r$, it is in the distance x from the internal end of the tube. So the flow will be viscous-molecular in a surrounding of this point $(x-d, x+d)$. We can consider the flow in the part $(x-d, x)$ as molecular flow and in the part $(x, x+d)$ as viscous flow, but the result will be not exact. We derive the pressure p_x from the relation for mean-free-path λ of molecule

$$\lambda = \frac{kT}{p\sqrt{2}\pi d_0^2} \ , \tag{31}$$

where d_0 is the diameter of the molecule. For air at $T = 298.15K$ Eqn.(17) gives:

$$\lambda \approx \frac{6 \cdot 10^{-3} Pa \cdot m}{p} \ . \tag{32}$$

If we substitute $\lambda = 2r$ and $p = p_x$, we obtain

$$p_x = 3 \cdot 10^{-3} Pan \cdot m \ \frac{1}{r} \ . \tag{33}$$

Now we know the pressure in the interface between molecular and viscous flow. It is necessary to determine its position, which is characterized by the variable x. We determine this value by vacuum conductance in both parts of the tube.

According to (3) and (16) the conductance of the part with molecular flow is

$$C_{0x} = \frac{8}{3}\frac{r^3}{x}\sqrt{\frac{\pi kT}{2m}} \ . \tag{34}$$

After introducing a new constant $a_{0x} = \frac{8}{3} r^3 \sqrt{\pi kT/(2m)}$ we obtain

$$C_{0x} = a_{0x}\frac{1}{x}. \tag{35}$$

According to (3) and (11) the conductance of the part with viscous flow is

$$C_{xl} = \frac{\pi r^4}{16\eta}\frac{1}{l-x}(p_l + p_x) \ . \tag{36}$$

After introducing again $a_{xl} = \pi r^4/(16\eta)$ we obtain

$$C_{xl} = \frac{a_{xl}}{l - x}(p_l + p_x) \ . \tag{37}$$

In a stationary state the flow Q through any cross-section of the tube is constant. It means

$$Q_{0x} = Q_{xl} \ , \tag{38}$$

where Q_{0x} is the flow in the part $0 - x$ and Q_{xl} in the part $x - l$. If we substitute Q_{0x} from (21), Q_{xl} from (23) and take into account Eqns. (3),(11) and (16), we get

$$\frac{a_{0x}}{x}(p_x - p_0) = \frac{a_{xl}}{l - x}(p_l + p_x)(p_l - p_x) \ . \tag{39}$$

The solution of this equation is

$$x = l \frac{a_{0x}(p_x - p_0)}{a_{xl}(p_l^2 - p_x^2) + a_{0x}(p_x - p_0)}. \tag{40}$$

The parameter p_x can be determined from Eqn. (19). Other parameters depend only on the tube geometry, a kind of flowing gas and its temperature. It is possible to determine the value x and the flow of gas Q from Eqns. (11) or (16). These equations have been derived on the assumption that pressure differences are constant. In fact pressure in the cathode-ray tube will increase very slowly. Parameters of the cathode-ray tube will be deteriorated at the pressure 10^{-2} Pa, which is still negligible in comparision with athmospheric pressure. We can consider the difference of pressures and the flow of air as constant. The pressure inside the cathode-ray tube will increase linearly with time.

The flow is defined according to (2), as a quantity of gas G in units pV per time unit

$$Q = \frac{G}{t} = \frac{pV}{t},$$

then

$$G = pV = Qt$$

is the quantity of gas, which arrives in the cathode ray tube in time t. To determine the quantity G_t of air in the cathode-ray tube in time t, it is necessary to add the initial quantity of air

$$G_0 = p_0 V \ .$$

We obtain time dependence of the quantity of air

$$G_t = p_t V = p_0 V + Qt$$

and of the pressure p inside the cathode-ray tube

$$p_t = p_0 + \frac{Q}{V} t \ . \tag{41}$$

The dependence (27) is presented in Fig. 3.

Figure 4: Pressure in the cathode-ray tube versus time.

11 Numerical calculations

The pressure existing in all kinds of cathode-ray tubes is approximatelly $10^{-4} - 10^{-3} Pa$, 10^{-2} Pa is a limit value, at which parameteres of the tube will not decrease under the "critical value". If the pressure increases by one order of magnitude (it means $0.1 Pa$), a glow discharge will arise in the space of the cathode-ray tube and the cathode-ray tube will not work.

We choose the classic color cathode-ray tube with the diagonal of 63 cm and with the deflection angle of 90°. The volume of such cathode-ray tube is approximatelly 20 dm^3.

We use constants:

Boltzmann constant	$k = 1.38 \times 10^{-23} JK^{-1}$,
Ludolf number	$\pi = 3.14159$,
air viscosity	$\eta = 0.018 \times 10^{-3} Nsm^{-2}$,
mass of the "air molecule"	$m = 4.81 \times 10^{-26} kg$.

Assumed parameters:

volume of the cathode-ray tube	$V = 0.02$ m^3,
initial pressure	$p_0 = 10^{-3}$ Pa,
atmospheric pressure	$p_{\mathrm{a}} = 101 \times 10^3$ Pa,
length of the hole (tube)	$l = 1.5 \times 10^{-2}$ m ,
radius of the hole (tube)	$r = 5 \times 10^{-7}$ m,
temperature	$T = 298.15$ K.

From these values we obtain for intermediate pressure according to (19)

$$p_x \simeq 6000 \text{ Pa}$$

and values of constants acording to (20) and (22):

$$\begin{aligned} a_{0x} &\simeq 1.22 \times 10^{-16} m^4 s^{-1}, \\ a_{xl} &\simeq 6.8 \times 10^{-22} m^5 kg^{-1} s \ . \end{aligned}$$

From (26), with $p_l = p_{\mathrm{a}}$,we obtain

$$x \simeq 1.4 \times 10^{-3} m = 1.4 mm.$$

It is easy to determine the value of gas flow from (11) with $l - x$ and from (16) with x in palce of l:

$$Q \simeq 5 \times 10^{-10} m^3 Pa s^{-1} \ .$$

The pressure in the cathode-ray tube after about two hours (that is the duration of a match) is according to (27) with $t = 7200s$:

$$p_t \simeq 1.18 \times 10^{-3} Pa \ .$$

This value differs very little from the initial pressure and we can say that *the cathode ray tube will stay functional and we can watch a football match without any problems till the end*.

Two additional processes are important during the flow of the air into the cathode-ray tube. One of them is capillary condensation. This effect appears when there is higher humidity of air in the room.

The second process is connected with a production technology. A thin film of getter with a large surface and great adsorbing ability is produced on the internal side of the cathode-ray tube. Getter adsorps gases emitted during the heating up of the cathode and also a little gas, which penetrates through the microscopical holes into the cathode-ray tube.

We need not take into account these effects (plugging of the hole by a condensate and adsorption effects of the getter), because both of them prolong the working time of the cathode ray tube.