prob11.tex **Problem 11**

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Introduction

To set paper on fire, the best way to do this is making an image of the Sun or the Moon with an optical system. All other ways decrease the intensity of the image on the paper and so the temperature of the paper.

In our solution we first calculate the intensity of the image to reach the burning temperature of paper. Then we show that a normal lens makes a solar image of an intensity that is much higher. So the paper can be set on fire with solar radiation.

In the next part we show that lunar radiation is to weak to do this. Then we proof that a combination of more than one lens will not help. So lunar radiation can not set paper on fire.

Solution

General data

The burning temperature of paper is $T' \approx 550K$. And the apparent angular diameter of the Sun and the Moon on the sky is $x = 31\,'$.

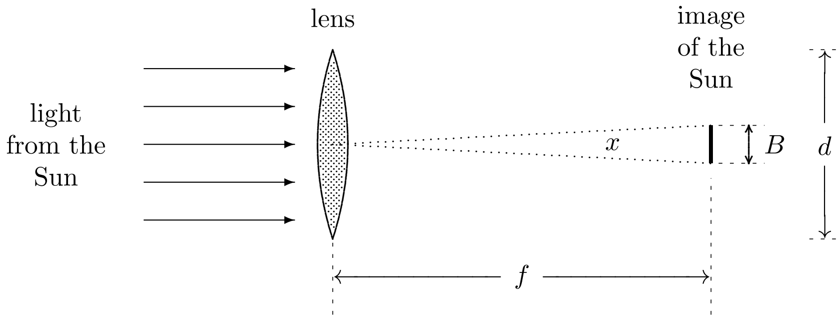
If we make an image of the Sun with a lens, it will have an intensity I. To set a paper (surface A) on fire, it has to have the temperature T'. Then the incoming power $P_{\text{in}} = IA$ must be greater then the outgoing power (black-body radiation of the hot paper) $P_{\text{out}} = 2\sigma A(T'^4 - T_0^4)$ (σ — Stefan-Boltzmann constant, normal temperature $T_0 \approx 300K$). This gives the condition:

$$I > 9500\,\frac{\text{W}}{\text{m}^2}\ . \qquad (1)$$

An image intensity on the paper has to be greater than this value to set the paper on fire.

Image of the Sun with a single lens

Now we try to set the paper on fire by making an image of the Sun with a single lens.



This image will be in the focus of the lens because of the long distance to the Sun. To get the intensity I of the image, we need the following equations:

the intensity of solar radiation on the Earth's surface $I_0 = 1400\,\frac{\text{W}}{\text{m}^2}$,

image diameter $B = xf$,

conservation of power $I_0 \frac{\pi d^2}{4} = I \frac{\pi B^2}{4}$.

This leads to

$$I = \frac{I_0 d^2}{x^2 f^2} \ . \qquad (2)$$

With a good lens ($\frac{f}{d} \approx 2$) we get an image intensity of

$$\boxed{I \approx 4.6 \times 10^6\,\frac{\text{W}}{\text{m}^2}}$$

This is enough to set the paper on fire.

Image of the Moon with a single lens

With lunar radiation we now try the same. We use a single lens. The derivation is exactly the same as for the Sun. But now the intensity of lunar radiation is

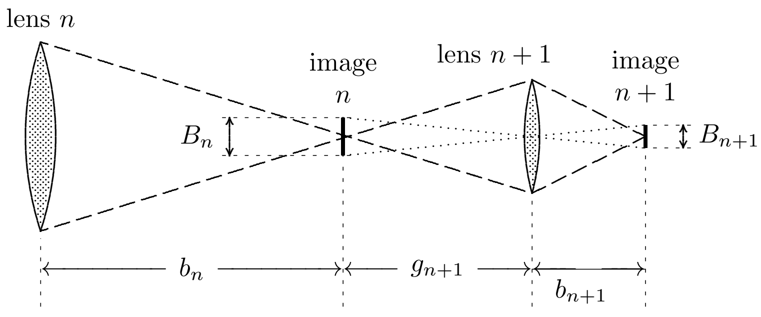
$$I_0 = 0.0014\,\frac{\mathrm{W}}{\mathrm{m}^2}\ .$$

With (2) we get

$$\boxed{I \approx 4.6\,\frac{\mathrm{W}}{\mathrm{m}^2}}$$

This is not enough to set the paper on fire. But what will we get, if we use more then one lens, if we make another image of the first image, and so on? So we examine a combination of lenses.

Two lenses out of a combination



The intensities of the two images are I_n and I_{n+1}. The diameters of the two lenses are d_n and d_{n+1} respectively.

Relation between the two image diameters: $\dfrac{B_{n+1}}{b_{n+1}} = \dfrac{B_n}{g_{n+1}}$.

If we do not want to waste energy (no loss), the diameter d of the following lens $n+1$ has to be in the whole light cone of the lens n, thus $\dfrac{d_n}{b_n} = \dfrac{d_{n+1}}{g_{n+1}}$.

Conservation of power: $I_n \dfrac{\pi B_n^2}{4} = I_{n+1} \dfrac{\pi B_{n+1}^2}{4}$.

So we get

$$I_{n+1} \frac{b_{n+1}^2}{d_{n+1}^2} = I_n \frac{b_n^2}{d_n^2} \ ,$$

what means

$$\boxed{I \frac{b^2}{d^2} = \text{constant}}$$

So the term Ib^2/d^2 is always constant no matter how many lenses are used. To get this term, we take our equation for a single lens (our first lens in the combination).
For the first lens we have (see (2)):

$$I = \frac{I_0 d^2}{x^2 f^2} \ , \qquad (3)$$

$$I \frac{b^2}{d^2} = \frac{I_0}{x^2} \ .$$

This term will be constant with each following lens. But we get the maximum intensity I with the minimum b/d, and this is for the best lenses about 2 or more because b must be bigger then the focal length f. Then we get a maximum intensity with a combination of lenses:

$$I = \frac{I_0 d^2}{x^2 b^2} \ , \qquad (4)$$

$$\boxed{I \simeq 4.6 \ \frac{\text{W}}{\text{m}^2}}$$

This is the same intensity as with a single lens. It is not possible to set a paper on fire only with lunar radiation and lenses. A combination of more lenses will not help us. Especially when we take into account that this intensity is an upper limit because of losses (lens, reflection of the Moon's surface etc.) and wave optics.

How should the Moon be alike?

Let us calculate the intensity I of the image on the paper as a function of the Moon's parameters. We use the following values:

intensity of solar radiation I_{solar},
radius of the Moon R,
distance of the Moon from the Earth D,
albedo (fraction of reflected light) n.

Now, the intensity of the reflected lunar radiation on the Earth is

$$I_0 = \frac{nI_{\text{solar}}}{2} \cdot \frac{R^2}{D^2} \ .$$

And with the apparent lunar diameter $x = 2R/D$ we get

$$I_0 = \frac{nI_{\text{solar}}}{8} x^2 \ .$$

And with (4):

$$I = \frac{I_0 d^2}{x^2 b^2} = \frac{nI_{\text{solar}}}{8} \cdot \frac{d^2}{b^2} \ .$$

We can see that the intensity I of the image on the paper only depends on the lunar albedo n. So the Moon should have a higher albedo. But even with $n = 1$ we would only get an intensity of

$$\boxed{I_{\max} \simeq 46 \, \frac{\text{W}}{\text{m}^2}}$$

Too less to set the paper on fire.

Summary

As we shown, it is possible to set paper on fire with solar radiation (this can be proved with a very easy experiment) but not with lunar radiation. Even with a combination of lenses.

Of course, one could have an idea to use other optical systems. It is clear that the best systems are those that make images. So the other possibility to lenses are concave mirrors. But for mirrors we have to use the same equations as for lenses. So we need a mirror with a d/f of much more than 25.

Calculations prove that a parabolic mirror with such a d/f and a focal length of only $0.1m$ would be longer than $4m$. And with a more realistic d/f of 50, because of losses, we even get a length of $15m$. Such length are of course unrealistic and wave optics begin to play an important role here.