Problem 13

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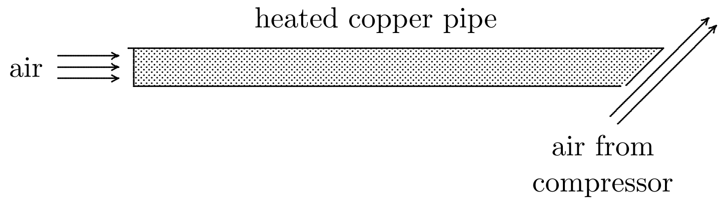
Introduction

The task of this problem is to construct an air lens. To change the index of refraction of air you can change either the pressure or the temperature of the air. It is much easier to change the temperature then the pressure.
In the solution I derive an equation for the focal length of a long air lens.
The experiment shows how well the theory fits to reality.

Solution

The air lens



We used a heated copper pipe (inner radius $R = 0.004\,\mathrm{m}$ and length $L = 0.4\,\mathrm{m}$). One of its ends was cut to about $45\,^\circ$. A compressor blew air by this end, so air was sucked in from the other end continuously and was heated by the copper pipe (ΔT hotter then the room air).

Temperature function

First we have to have the temperatures in the pipe. We can assume a parabolic temperature function, because the inner parts of the air can't be heated as much as the outer parts (air in the middle has got higher velocity). So a parabolic function is a good approximation. And the experiment (the air lens **does** focus) shows that this is a good approximation.

$$T = T_0 + \frac{\Delta T}{R^2} r^2 \qquad (1)$$

The index of refraction of air fulfilles the following equotation:

$$\begin{aligned} n - 1 &\sim \rho \\ n - 1 &\sim \frac{1}{T} \\ (n-1)T &= \text{const.} \end{aligned}$$

With the index of refraction n_0 at the normal temperature T_0:

$$\begin{aligned} (n-1)T &= (n_0 - 1)T_0 \\ n - 1 &= \frac{(n_0 - 1)T_0}{T} \\ n &= 1 + \frac{(n_0 - 1)T_0}{T} \end{aligned}$$

With (1):

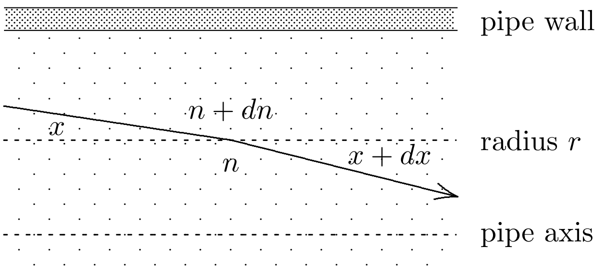
$$n = 1 + \frac{(n_0 - 1)T_0}{T_0 + \frac{\Delta T}{R^2} r^2}$$

$$\frac{dn}{dr} = -\frac{(n_0 - 1)T_0}{\left(T_0 + \frac{\Delta T}{R^2} r^2\right)^2} \cdot \frac{\Delta T}{R^2} 2r$$

$$\frac{dn}{dr} = -\frac{2r\Delta T(n_0 - 1)}{T_0 R^2} \qquad (2)$$

The light beam in the pipe

How does a light beam travels through the heated air in the pipe?



We can write the law of Snell as:

$$(n + dn)\cos x = n\cos(x + dx)$$

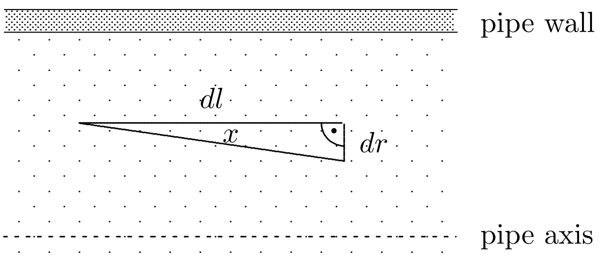
So we will get with $\cos z \approx 1 - z^2/2$ (beam is nearly horizontal):

$$\begin{aligned} (n + dn)(1 - x^2/2) &= n(1 - (x + dx)^2/2) \\ n - nx^2/2 + dn - dn\,x^2/2 &= n - nx^2/2 - nx\,dx - n\,dx^2/2 \\ dn - dn\,x^2/2 &= nx\,dx - n\,dx^2/2 \\ dn\,(1 - x^2) &= -nx\,dx\,(1 + dx/2x) \end{aligned}$$

We can neglect the x^2 and $dx/2x$. So we get:

$$dn = -nx\,dx \qquad (3)$$

Relation between dr, dl and the angle x



If the light beam travels a small way dl its radius (distance to the pipe axis) will decrease with the value dr. This equation is only the approximation of the tan for small angles (here x):

$$x = \frac{dr}{dl}$$

With equation (3):

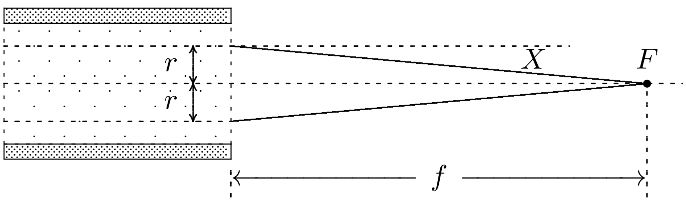
$$\frac{dn}{dr}\,dl = -n\,dx$$

Now we just integrate both sides and we get the end angle X of a light beam that started horizontal and travells through a lens of the length L'. This length is the **effective** length of the air lens, which is shorter then the real length L of the copper pipe, because the parabolic temperature function is only an average function (the air will be heated during its way through the copper pipe).

$$\frac{dn}{dr}\int_0^{L'} 1\ dl = -n\int_0^{X} 1\ dx$$

$$\frac{dn}{dr}\frac{L'}{n} = -X \qquad (4)$$

Relation between X, r and the focal length f



With the small rectangular triangle we get the needed relation:

$$\frac{r}{f} = X$$

And if we use our equations for $\frac{dn}{dr}$ (2) and X (4) we get the focal length f:

$$\boxed{\frac{1}{f} = \frac{2\Delta T(n_0 - 1)L'}{T_0 R^2 n}}$$

Experiment

In our experiment we used a air lens with the following data:
$\Delta T = 42\,\mathrm{K}$
$n_0 = 1.0003$
$L = 0.4\,\mathrm{m}$
$T_0 = 296\,{}^\circ\mathrm{K}$
$R = 0.004\,\mathrm{m}$
In this experiment we get a focal length of about:

$$\boxed{f_{exp} \approx 1.65\,\mathrm{m}}$$

With the theory we get with the given data:

$$\boxed{f_{theor} = 0.47\,\mathrm{m}}$$

If we consider that the effective length L' is shorter than L, we can assume that the theory fits quite well to the experiment ($L \approx 4L'$).

Summary

This construction has small angles between the light beams and the layers of different index of refraction. The alternative construction would be a normal
lens-like one. But in this case we would have a focal length of over 130 m with the same temperature difference as above and a lens radius of only 1 cm. So the long air lens is the best construction.
Such air lenses are quite complicated and don't seem to have real application.
But this air lens construction can be important in future to focus high intensity laser beams. In such a case only air lenses could resist the high temperatures because of the energy losses.