Problem No.10 – Water Dome

Solution:

In this problem we are supposed to explain and describe the phenomenon called water dome. We solved this problem both experimentally and theoretically.

Now let s describe the experiment.

We used a water tap as a source of water jet. This has some advantages:

Firstly the experimental device is very easy to make up.

Secondly we can change the velocity of the water by moving the bar up and down.

Thirdly we can change the rate of water flow.

Fourthly we can change the surface tension by changing the temperature of water.

We conducted the experiment with different parameters, and observed the water dome. We took photos of several of the experiments. There are photos and a table with corresponding values in the appendix. We found out that the shape of water dome depends on the following parameters:

If the velocity of water increases, the water dome gets larger. For low velocities the dome is closed. If the velocity is too large the dome cannot close and sprays into drops.

The same will happen if we increase the rate of water flow or decrease the surface tension by increasing the temperature.

If the radius of the bar increases, the velocity of water decreases and the dome gets smaller. If the radius decreases the radius of the water jet is no longer negligible and the water doesn't flow horizontally from the top of the bar.

The dome shape also greatly depends on the pressure inside the dome. If the pressure is above atmospheric pressure, the dome gets larger, if the pressure is bellow atmospheric pressure, the dome gets smaller. Thus the dome shape depends also on its history, which means the way it was produced. This possibility was not included into theoretical solution because it is not very frequent.

Now let us come to the theoretical solution. We are trying to determine a water dome shape, having provided the parameters so that the water dome is closed and does not spray into drops.

Secondly let us suppose that the radius of the bar is so small that we can neglect the decrease of the velocity on the top end of the bar, but, on the other hand, that the radius is big enough so that we could neglect the radius of the water jet.

Let us consider the element of water volume. Its coordinates in dependence on the time are determined by two unknown functions $x(t)$ and $y(t)$, which we wish to discover. The coordinates system is shown on Fig.18a.

On the water volume element acts the capillary pressure force F_k normal to its surface and the gravitational force vertically. As long as we want to write an equation of motion of the water element, we need to establish the force F_k, and in order to do this we need to determine the capillary pressure p_k.

For this we used the formula $p_k = \sigma . (c_1 + c_2)$, where c_1 and c_2 are the curvatures of two normal plane sections of the water dome, perpendicular to each other. So we need to conclude these curvatures. We chose the first section vertical. Second section is perpendicular to the first one and to the dome as well. The curvature of the vertical section can be easily found by using of the commonly known formula for curvature of a curve $y(x)$:

$c_1 = y''/(1 + y'^2)^{3/2}$ (derivations are by the x)

Now to the second section curvature. Firstly we need to find the equation of the second section. The coordinates system has the [0,0] point in the middle of the bar and horizontal z axis. The formula for $u(z)$ is

$$u = \pm\sqrt{y^2(x_0 + z\sin\alpha) - (y_0 - z\cos\alpha)^2}$$

where x_0, y_0 are coordinates of the intersection of the dome and the second section.

We'd like to express the curvature of curve $z = z(u)$ from it. We cannot express the curvature of the function $u(z)$, because the derivation of this function in $z = 0$ (which is the point we are interested in) is infinite. So as to do this we need to know the derivations of the function $z(u)$.
We used the formula for 1. and 2. derivation of inverse function:

For the function $y(x)$ there is $y' = 1/x'$ and $y'' = -x''/(x')^3$.

So we can easily find the first and second derivation of z by the u in $u = 0$, and so to find this expression of 2. section curvature:

$c_2 = 1/(y_0(y'_0 \sin\alpha - \cos\alpha))$, derivation is by x.

Now we know everything to establish capillary pressure p_k, and force F_k as well. We used formulas for derivations of composed function for we need x and y to be functions of time.

$$p_k = \sigma(y''x' - y'x'')/(x'^2 + y'^2) + \sigma/(y_0(\sin\alpha\, y'_0/x'_0 - \cos\alpha))$$

$$F_k = p_k \mathrm{d}S = p_k y\sqrt{x'^2 + y^2}\, \mathrm{d}\alpha\, \mathrm{d}t$$

Now we are ready with determining F_k. If we write 2 equations of motion for a water element (for each coordinate one) and substitute for F_k, we get the set of 2 differential equations for 2 unknown functions:

$$y''\mathrm{d}m = -F_k \cos\alpha \qquad x''\mathrm{d}m = F_k \sin\alpha + g\,\mathrm{d}m$$

where $\mathrm{d}m = f/2\pi\, \mathrm{d}t\, \mathrm{d}\alpha$ is mass of the element of volume and f is rate of water mass flow (f = mass of the water, which flows of the water tap/corresponded time).

This set cannot be solved analytically but we solved it very easily numerically.

For the numerical solution we can approximate the functions by very many small parabolic arcs. (On each of these arcs the 2nd derivations of x and y by the time are constant). We know initial values of x, y and all derivations. From Taylor's expansion we can get the values of x, y and the first derivations on the next arc from the ones on the last one. Let us mark

$t_i = (i - 0.5)t$, where t is sufficiently small time interval

$x_i = x(t_i)$

$y_i = y(t_i)$

$x'_i = x'(t_i)$,

and so on. The i-th parabolic arc is between times $t_i - 0.5 \, . \, t$ and $t_i + 0.5 \, . \, t$ (see Fig.18c). Taylor expansion (x'' and y'' are constant implies that $x''' = y''' = 0$!):

$x_i = x_i + x'_i t + 0.5 x''_i t^2$

$y_i = y_i + y'_i t + 0.5 y''_i t^2$

$x'_i = x'_i + x'_i t$

$y'_i = y'_i + y'_i t$

Now we use the differential equations - we take them as a set of two linear equations for two unknown – x'' and y''. So we can easily express x''_i and y''_i from x_i, y_i, x'_i, y'_i by using the Cramer's law. So we can find out all necessary values from the last ones. Because the function, we are looking for, and their derivations are continuous, we may claim that our method is correct, which means that if we choose t (and so the sizes of parabolic arcs) small enough, we get a solution within desirable tolerance. We used computer to establish this numerical solution, and draw the water dome shape for some parameters. One of the dome shapes is shown at Fig.18d.

Now let's come to the conclusion. We found the numerical solution for different values of parameters, and compared them with observed shapes of the water domes. We found excellent agreement between the theoretical model and the experiment, and this verified that our model was valid.

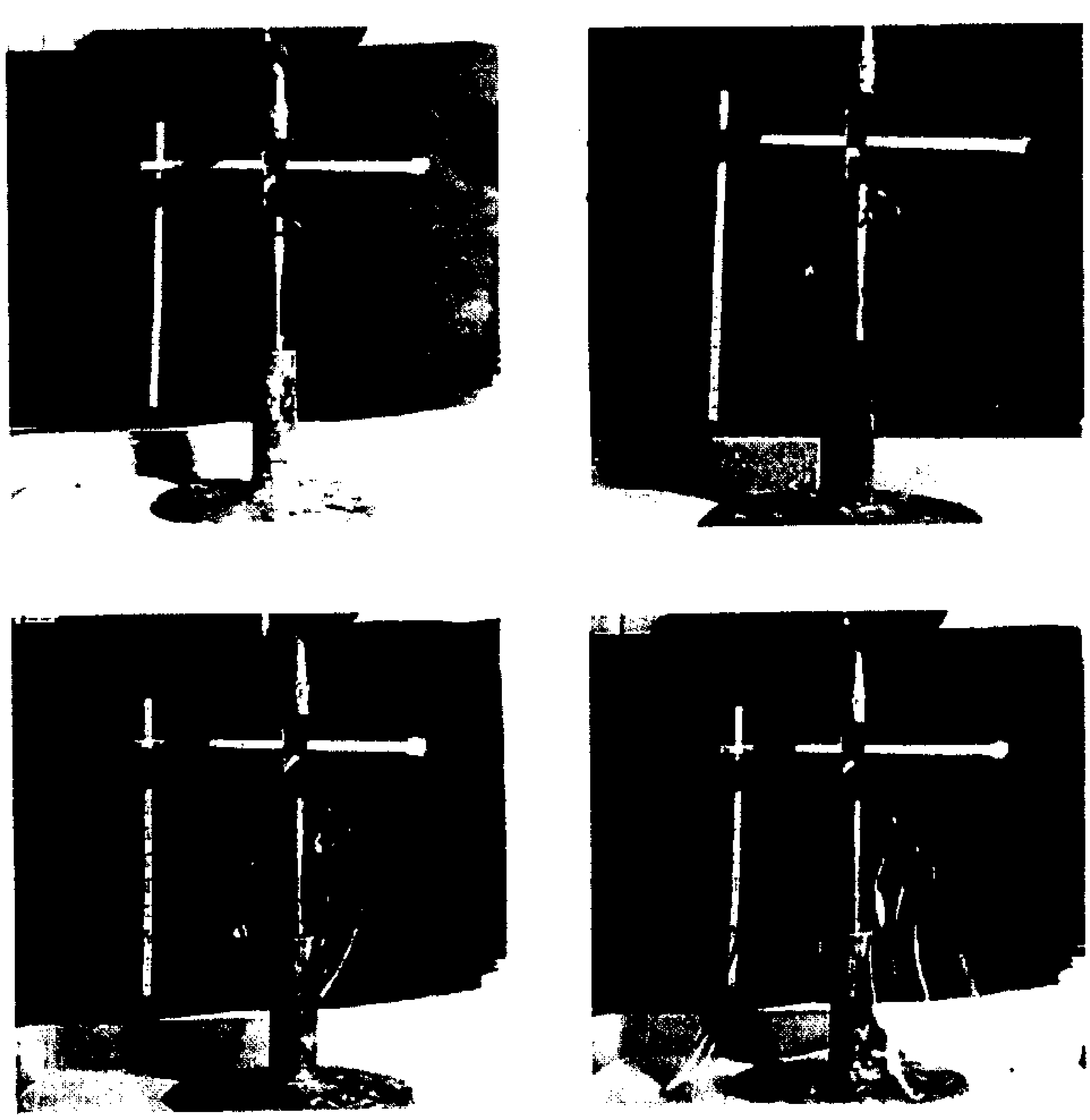

Fig.18(a-d) Various types of water domes