A toy (called "Levitron") consists of a magnetic spinning top and a plate containing magnets. The top may levitate above the magnetic plate. Under what conditions can one observe the phenomenon?
Presentation Plan

1. **Description of LEVITRON**
   - Levitron components;
   - Experiment; Description of Levitron action (Phenomenology);
   - Dependence on environment.

2. **Physics of Levitron**
   - Qualitative analysis;
   - Quantitative analysis; Basic assumptions;
   - System Energy analysis - Equilibrium condition, Stability condition;
   - Top's spin upper limit.

3. **Equilibrium area**
   - Theoretical calculation of base magnetic field;
   - Measured field;
   - Area of equilibrium.

4. **Conclusion**
Description of Levitron

1. **Spinner (Top)** – nonmagnetic spindle inserted in a flat, toroidal, permanent magnet (Ceramic)
2. **Base** – relatively large toroidal ceramic permanent magnet.
3. **Plastic lifter plate**

Between Base and Top acts **dipole-dipole repulsion** force

Magnetic Top levitates above the Base due to **gradient** of base magnetic field.
Description of Levitron action (Phenomenology)

- Top rotation speed:
- Top precession speed:
- Top “oscillation” frequency:

\[ \Omega_{\text{spin}} \approx 35\, \text{Hz} \]
\[ \Omega_{\text{precession}} \approx 5\, \text{Hz} \]
\[ \Omega_{\text{lateral}} \approx 1\, \text{Hz} \]

\[ \Omega_{\text{spin}} \gg \Omega_{\text{precession}} \gg \Omega_{\text{lateral}} \]

- Spin precesses around the local direction of the field and Magnetic moment of Top effectively on the average points antiparallel to the local magnetic field lines.
- Base magnetic field lines are not vertically directed. Their direction is varying.
The magnetic field lines
Sensitivity to environment

Top Levitation is influenced by:

• External magnets or magnetized bodies,
  [video]

• Iron bodies;
  Placing the base on thick iron plate changes the shape and value of magnetic field of base.
  Equilibrium point is lower while top's mass must be larger.

• Temperature fluctuations
  One needs to change top's mass when the temperature of environment changes.
The magnetic field lines
Physics of Levitron

Qualitative analysis
For the stable levitation two conditions must be satisfied:

a) Top's weight must be balanced by magnetic Repulsion force
b) In the levitation point potential energy must have minimum.

Ordinary spinner precesses around vertical gravitational field lines.

Magnetic top precesses around local magnetic field lines (which in different points have different directions).

Due to gyroscopic effect top's precession axis always tries to be aligned on average to the direction of local magnetic field line.

At radial excursion top's precession axis tries to "reorient" to magnetic field local line direction.

Reorientation energy of precession axis creates the potential well.
Quantitative analysis; Basic Assumptions

For our further investigation let us make such assumptions:

a) In the magnetic field of base there are no currents and magnetic charges, i.e. Maxwell equations take the form:

- **Gauss law** for magnetic field: \( \vec{V} \cdot \vec{B} = 0 \) – magnetic charge does not exist;
- **Amper's law:** \( \vec{V} \times \vec{B} = \mu_0 \vec{j} + \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t} \) - electrical current and electric field variation create magnetic field. Since on our case there are no electric currents and electric field \( \vec{V} \times \vec{B} = 0 \)

b) Top is Magnetic Dipole and its magnetic dipole centre coincides with Mass centre;
- Top is „Fast“, precession angle is small - the angular momentum is along the spin axis of the top which also coincides with the magnetic moment axis;
- Top's precession frequency is much larger than lateral oscillations frequency \( \Omega_{\text{precession}} \gg \Omega_{\text{lateral}} \)
- Top's magnetic moment \( \vec{M} \) has constant value \( \vec{M} = |\vec{M}| = \text{const} \)
**Quantitative analysis; Potential Energy of Magnetic Spinner**

The force exerted by the magnetic field $\vec{B}$ on a dipole of moment $\vec{M}$ in air or vacuum depends on the **directed gradient** of the field (Сивухин. т.3, стр.243):

$$\vec{F}_m = (\vec{M} \cdot \vec{\nabla}) \cdot \vec{B}$$  \hfill (1)

The potential energy of the top (since in average $\vec{M} \parallel \vec{B}$)

$$U = - (\vec{M} \cdot \vec{B}) + mgz \approx MB + mgz$$  \hfill (2)

Representing $\vec{B}$ as a **Taylor series** in the vicinity of levitating point and taking into account Maxwell equations we get:

$$B_z = B_0 \left[ 1 + \alpha_1 z + \alpha_2 \left( z^2 - \frac{1}{2} r^2 \right) + O((r,z)^3) \right]$$

$$B_r = B_0 \left[ - \alpha_1 \frac{r}{2} - \alpha_2 zr + O((r,z)^3) \right]$$  \hfill (3)

Where

$$\alpha_1(z_0) \equiv \left[ \frac{1}{B_z} \frac{\partial B_z}{\partial z} \right]_{z=z_0=0; r=0}$$ \hspace{1cm} ; \hspace{1cm} $$\alpha_2(z_0) \equiv \frac{1}{2} \left[ \frac{1}{B_z} \frac{\partial^2 B_z}{\partial z^2} \right]_{z=z_0=0; r=0}$

In this approximation the potential energy of top is: $[1,2]$

$$U \approx B_0 + \left\{ \frac{mg}{\mu} + B_0 \alpha_1 \right\} z + B_0 \alpha_2 z^2 + \frac{1}{2} B_0 \alpha_2 \left\{ \frac{1}{4} \alpha_2^2 - 1 \right\} r^2 + \ldots$$  \hfill (4)
Let us investigate this expression:

\[
U \approx \left[ B_0 + \left( \frac{mg}{\mathcal{M}} + B_0 \alpha_1 \right) z + B_0 \alpha_2 z^2 + \frac{1}{2} B_0 \alpha_2 \left( \frac{1}{4} \alpha_1^2 - 1 \right) r^2 + \ldots \right]
\]

(4)

a) In the levitating point must be energy extremum:

\[\frac{\partial U}{\partial z} = 0.\]

**Gradient** of base magnetic field balances gravitational force (**Equilibrium condition**):

\[\alpha_1 = \left[ \frac{1}{B_0} \frac{\partial B_z}{\partial z} \right]_{z=z_0=0; r=0} = -\frac{mg}{\mathcal{M}B_0} \]

(5a)

\[\frac{\partial^2 U}{\partial z^2} > 0; \quad \frac{\partial^2 U}{\partial r^2} > 0\]

b) For stable equilibrium **energy** must have **minimum**: i.e. coefficients of \( z^2 \) and \( r^2 \) must be positive (**Stability condition**):

\[\alpha_2 > 0; \quad \frac{1}{4} \alpha_1^2 - 1 > 0 \quad ; \quad \alpha_1^2 > 4 \alpha_2 \]

(5b)

\[\frac{1}{4 \alpha_2} \alpha_1^2 \]

(5c)
Top Spin Frequency Upper Limit

Trapping conditions set a restriction on the top spin frequency upper limit: \[1\]

\[
\omega \leq \frac{1}{r_{\text{eff}}^2 \cdot g} \left(\frac{M}{m} \cdot B_0\right)^{3/2}
\]

The top spin frequency must be less than this value, to have the levitation.

If the top is "too fast", the precession frequency will be too slow to allow the top to reorient to the local field direction as the top makes its radial excursion.
Equilibrium area

Assume base as a consequence of vertically oriented magnetic dipoles stacked as a horizontal ring.

Similarly to electrostatic problem the magnetic field along the ring axis will be: [2]

\[
B_z = \frac{\mu_0 M_{\text{ring}}}{4\pi R^3} \cdot \left\{ \frac{2 \left( \frac{z}{R} \right)^2 - 1}{\left[ \left( \frac{z}{R} \right)^2 + 1 \right]^{5/2}} \right\}
\]

Equilibrium area

\[ B_z \]

- ring dipole moment

\[ R \]
- Radius of base magnet (4.5 cm)

\[ z \]
- Distance from the base along axis.

When \[ z \approx R/\sqrt{2} \], \( B_z < 0 \);

When \[ z = R \sqrt{3}/2 \], \( B_z = 0 \);

When \[ z \rightarrow \infty \], \( B_z = \text{max} \);

When \[ z \rightarrow \infty \], \( B_z \rightarrow 0 \)

Here \( B_z \) is given in ( \( \times 10^{-4} \) Tesla);

\( z \) – in cm;

Magnetic constant \( \mu_0 = 1.257 \cdot 10^{-6} \text{ V·Sec/(A·m)} \);

Comparing (8) with measurements we get

\[ M_{\text{ring}} = 41.7 \text{ A·m}^2 \]
Calculated Equilibrium area

From (8) calculating relevant derivatives and using "Trapping Conditions" (5) we found the "Top Trapping Range".
For our case it is around 8 cm from the Base magnet.
Sensitive area

8 cm
Thank you for attention!

References:
Top Spin Frequency Upper Limit

Top motion equations (the \textit{torque} and \textit{force} equations):

\textit{torque} \quad \frac{d\vec{L}}{dt} = \vec{M} \Rightarrow \quad \text{[since angular momentum \quad \vec{L} = I\omega (\frac{\vec{M}}{I})]}

\frac{d\vec{M}}{dt} = \frac{\vec{M}}{I\omega} \times \vec{B} \quad (6a)

\textit{force} \quad m \frac{d^2 \vec{r}}{dt^2} = (\vec{M} \cdot \vec{V}) \cdot \vec{B} - mg\hat{z} \quad (6b)

$I = mr_{\text{eff}}^2$ - rotational Inertia of the Top.

From (6a)

\[ \omega_{\text{precession}} = \frac{MB}{I\omega} \]

the precession frequency is inversely proportional to the spin frequency.

Substituting in (6) $\vec{B}$ in Taylor series and imposing condition (5c) one can get [Simon et al. AJP 65 (4) 1997])

\[ \omega \leq \frac{1}{r_{\text{eff}}^2 \cdot g \left( \frac{m}{M} \cdot B_0 \right)^{3/2}} \quad (7) \]

If the top is "too fast", the \textit{precession} frequency will be \textit{too slow} to allow the top to reorient to the local field direction as the top makes its radial excursion.

This is \textit{the origin of the upper spin limit}.

\(\vec{M}\) of very fast top is oriented always vertically. Thus:

(a) – we can not use approximation (2).

(b) – in the force (1), (6) the radial component of magnetic field will not act on the top.
Earnshaw Theorem

Two conditions for stable levitation:

a) Balance of magnetic and gravitational forces;
b) Potential energy must have minimum in the levitation point.

Problem! „Earnshaw“ theorem:
If the interaction energy satisfies the Laplace equation:

\[ \nabla^2 U(\vec{r}) = \frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2} = 0 \]

It can not have the local minimum. It has only saddle points.

This concerns \( \frac{1}{r} \) - type potentials. Really, if \( f(r) = \frac{f_0}{r} \), (and magnetic interaction is such) then in spherical coordinates

\[ \Delta f \equiv \nabla^2 f \equiv \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial f}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial f}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 f}{\partial \phi^2} = \frac{1}{r^2} \frac{\partial}{\partial r} \left( -r^2 \frac{f_0}{r^2} \right) = 0 \]

Though it is for static case. We have DYNAMICAL stable equilibrium.