Task of 20th IYPT – example 3

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Slinky

Suspend a Slinky vertically and let it fall freely. Investigate the characteristics of the Slinky´s free-fall motion.

1.Analysis of the problem

What is slinky? Slinky [1] is an continuous spring system, Fig. 1, with uniformly distributed mass m and spring toughness k. For Slinky is typical that it extended itself because of its low spring
toughness. Naval engineer Richard James invented Slinky in 1945. Slinky is a typical toy for children, too. Object of our solution is physical description of Slinky´s free-fall motion (e.g.: amplitude, acceleration, velocity, air resistance...). First, we resume current information about slinky, then we observe free-fall motion (use camera). After then we want mathematically proof our observation (explain theoretical model) and write conclusions.

2.Observation

We suspend a slinky vertically and let it fall freely. As you can see on the pictures 3 **the lowest part not moved after dropping**. Our solution is based on this observation. There is damping oscillation. If the upper part crashes to the lower, transfer its momentum and there is going „miss out" energy (energy loss about 20%).

3.Skinkys used in our experiments

There are many materials, which we used to make a slinky (metal spring etc.). However we use only plastic Slinky. We work with 3 Slinkys with different length and spring toughness (165,130,60 cm).

Parameters

	Symbol	Slinky 1/2/3	Unity
Length	L_0	165/60/130	cm
Number of threads	N	39/20/40	
Mass	M	77/40/80	g
Material	Plastic		
Half of period	T/2	0,3/0,2/0,3	s
Frequency	f	1,67/2,5/1,67	Hz
Angular velocity	ω	3,33π/5π/3,33π	rad.s^{-1}

4.Before dropping

As you can see on Fig.2, there are different amplitude for individual thread.

$(l_1, l_2, \dots l_n)$ ….is set all members of arithmetical progression.

This material is homogenous – all threads have the same mass.

In equilibrium state:

$$F_k = F_g$$

$$kl_n = nmg$$

$$l_n = \frac{nmg}{k}$$

$$l = \frac{n}{2}(l_1 + l_n)\ l_n)$$

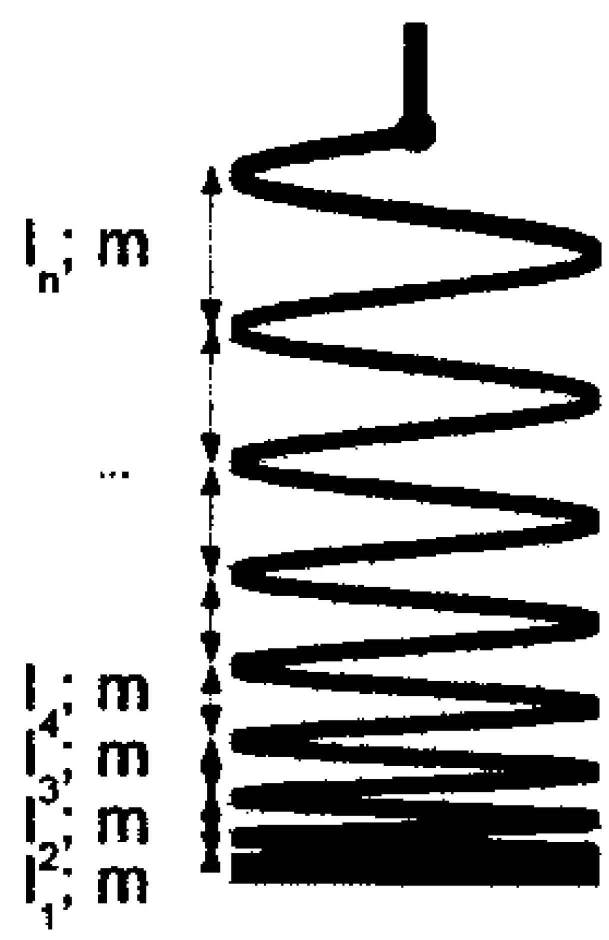

Fig. 1: Slinky´s threads

F_k	…force of elasticity
F_g	…gravity force
n	…numbers of threads
l	…length of slinky
l_n	…length of one of threads

Simplified slinky

We can simplify this spring by using concentrated parameters.

Two masses *m*, connected with mass-less spring length l_0 and spring toughness *k* and with its centre of masses *T* exactly in the middle of this system.

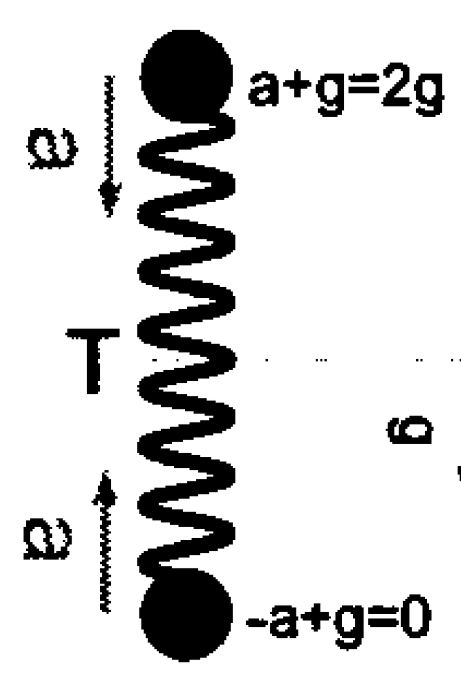

Fig.2: Simplified Slinky

5.Gedanken experiments

Imagine, that I am holding a slinky in an Einstein lift. And the lift is in a gravitational field. Now, the Einstein lift starts falling with an acceleration g and in the same moment I release the Slinky. So, the Slinky's centre of masses starts falling with *g* too and the Slinky's masses start to oscillate. We find out that both masses is moving with the same acceleration but opposite direction. Indicate acceleration, Fig. 3, of upper mass as au and acceleration of lower mass as ad. Next, we found out these accelerations. Now we have to add gravity acceleration to every point. (Ass you can see on the picture)

In lift we observe only the oscillation. However the is falling with gravitational acceleration g.

If we take one half of simplified Slinky and suspend it, we can say that the half of this spring has spring toughness doubled in comparison with original spring.

Of course, the deviation of half spring is half of delta l. From this result equilibrium:

$a=F_k/m$, $F_k = k\,u$

$$a_{\max}=\frac{2k}{m}\frac{\Delta l}{2}=\frac{k\Delta l}{m}=g$$

⇒Maximum acceleration of the lower mass equals with gravity acceleration but in opposite direction. Final acceleration of lower mass is zero. That means that the upper mass gives the lower its dynamics. This agrees with observation. Upper part move with acceleration equals with 2g.

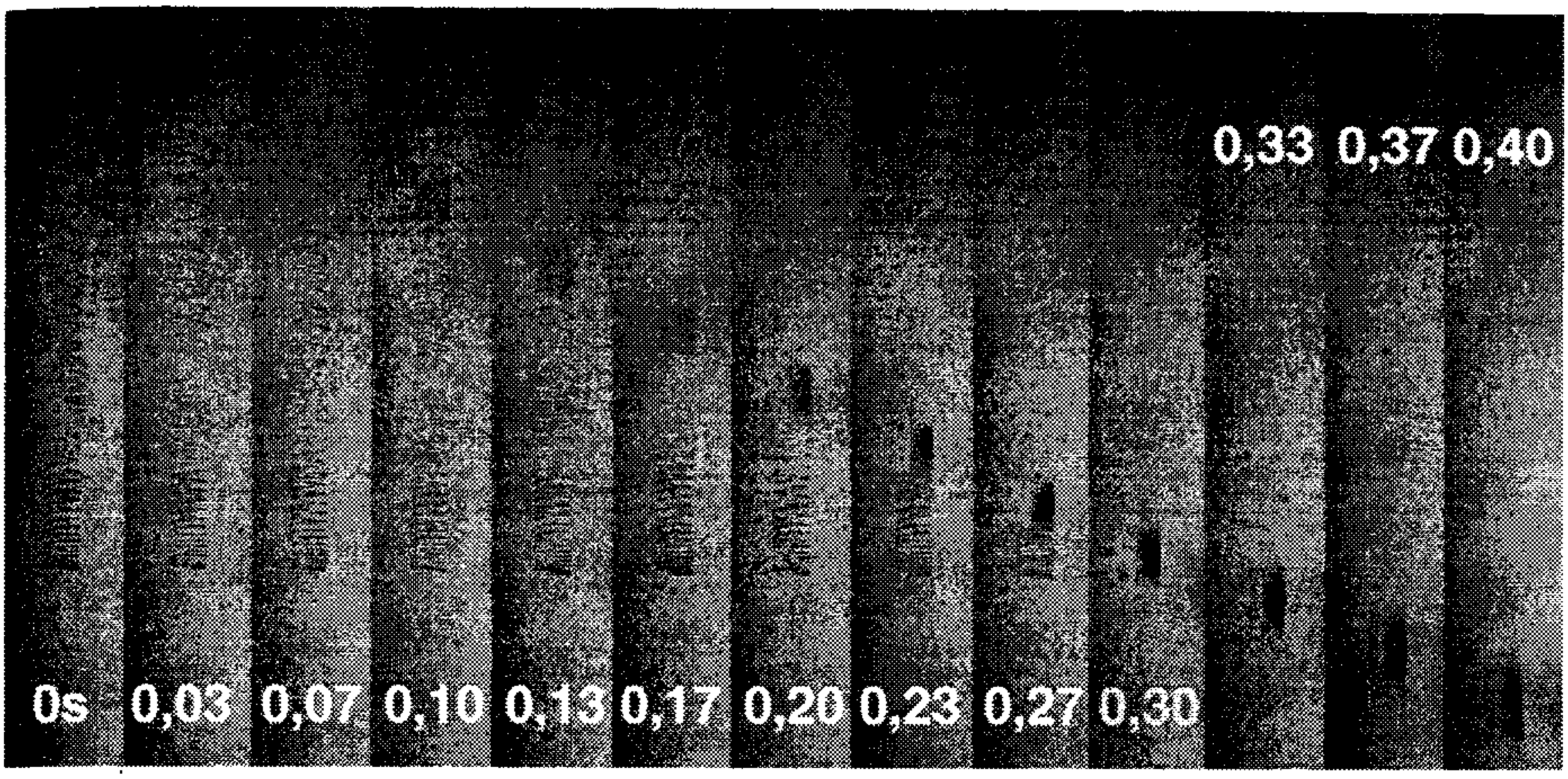

Fig. 3: experiments: Slinky 3

6.Mathematical proof – theory

For half of Slinky apply:

$$a = -\frac{2k}{m}u$$

$$a = \frac{dv}{dt} \quad \wedge \quad v = \frac{du}{dt} \quad \rightarrow \quad a = \frac{d^2u}{dt^2}(1)$$

$$F = F_k$$

$$ma = -2ku(2) \quad \wedge \quad 2k = K ... spring\ toughness\ of\ half\ Slinky$$

If we substitute (1) to (2):

$$m\frac{d^2u}{dt^2} = -Ku$$

$$\frac{d^2u}{dt^2} = -\frac{K}{m}u \quad \wedge \quad \frac{K}{m} = \omega^2$$

$$\frac{d^2u}{dt^2} = -\omega^2 u(t)$$

From this result:

$$u(t) = C_1 \cos \omega t + C_2 \sin \omega t(3)$$

To find out constants of integration C_1 and C_2 we have to know initial conditions then:

$$u(t) = \frac{\Delta l}{2} = \delta \qquad \rightarrow \qquad C_1 = \delta$$

$$v(0) = \frac{du}{dt}(0) = 0 \qquad \rightarrow \qquad C_2 = 0$$

C_1, C_2 → (3) → deviation relative to centre of gravity dependence on time:

$$u(t) = \delta \cos \omega t (4)$$

Velocity & acceleration equations:

$$v = \frac{du}{dt} = -\delta\omega \sin \omega t$$

$$a = \frac{dv}{dt} = -\delta\omega^2 \cos \omega t$$

Now we substitute amplitude of acceleration with g

$$a_{\max} = -\delta\omega^2 = \frac{mg}{2k}\frac{2k}{m} = g$$

Final equations:

$$v = -\delta\omega \sin \omega t (5)$$

$$a = -g \cos \omega t (6)$$

In acceleration equation we substitute time zero. Acceleration in this time is really –g.

7.Comparison - graph of dependence on time

If we calculate Slinky´s amplitudes dependence, Fig. 4, on time (lengths of Slinky after dropping) we use modified equation:

$$l(t) = \delta \cos 2\pi \frac{t}{T} + \delta = \delta\left(1 + \cos 2\pi \frac{t}{T}\right)$$

l ...Slinky´s amplitude dependence on time t

δ ...half of Slinky´s amplitude

T ...period-we establishes it from fraps. The accuracy is thousandth second.

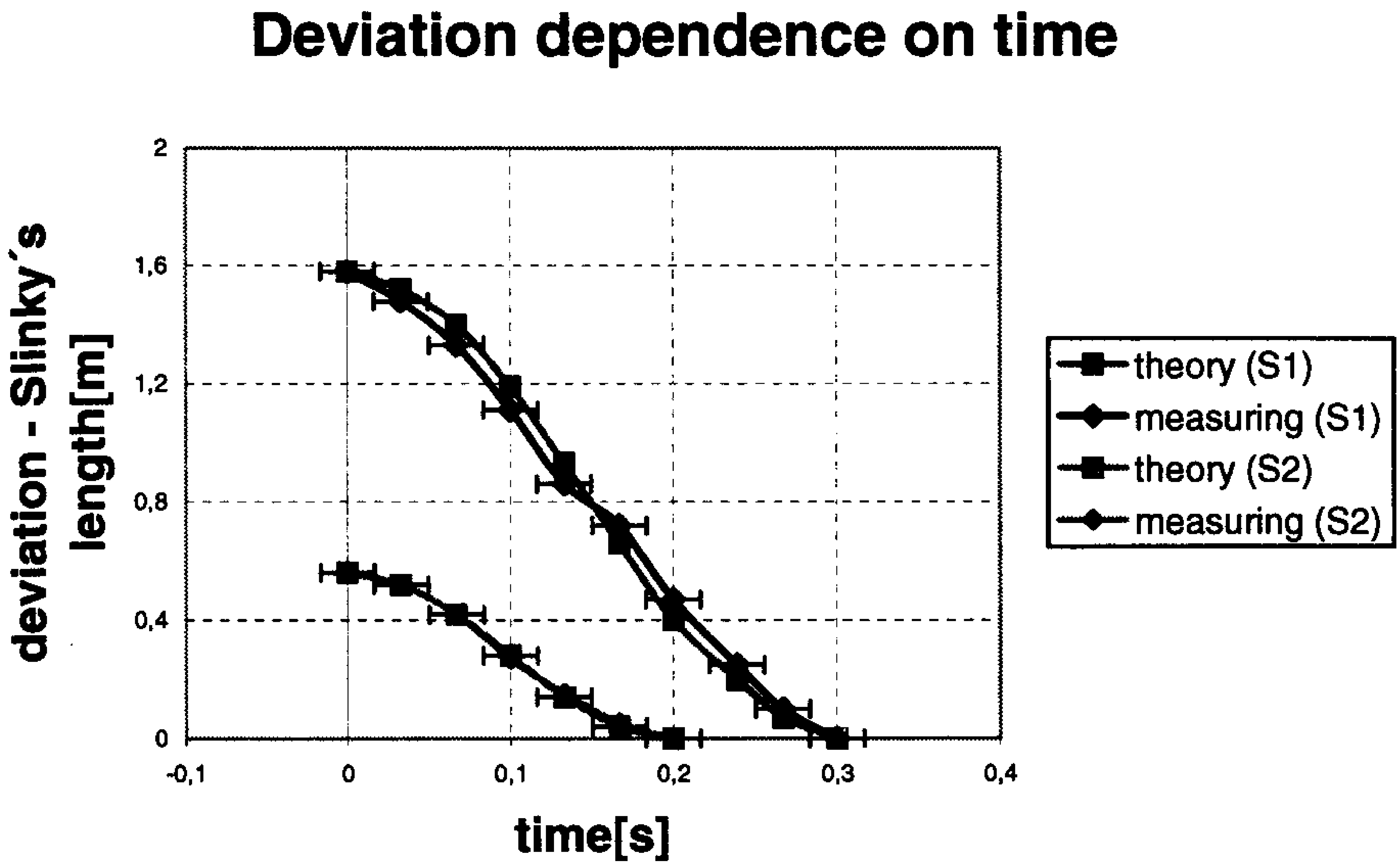

Fig. 4: deviation dependence on time: Slinky 1, Slinky 2

Damping oscillation

We measured the 60cm length Slinky. We observed, Fig. 5, the motion after crash

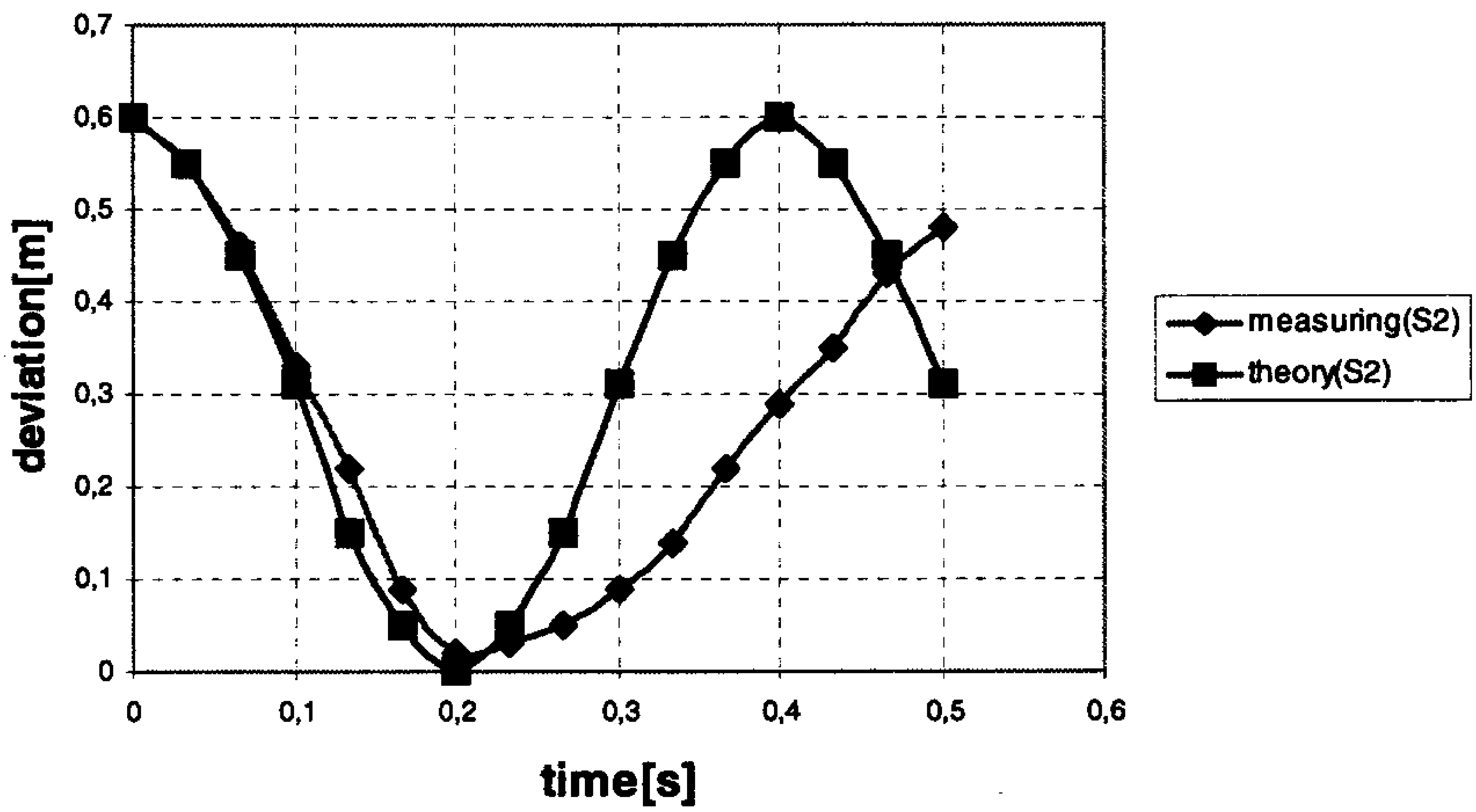

Fig. 5.: graph 1 Damping oscillation: Slinky 2

8.Comparison between slinky and classical spring

		SPRING	SLINKY
Deviation	y	$y_m\sin\omega t$	$\delta\cos\omega t$
Velocity	v=y´	$v_m\cos\omega t$	$-\delta\omega\sin\omega t$
Acceleration	a=y´´	$-a_m\sin\omega t$	$-g\cos\omega t$

Note: Symbol (´) express derivation

9.Conclusions

We **observe** and **measure this phenomenon with useless of digital camera.** On computer we measure deviation dependence on time in a special program (kamera-30fps). Then we mathematical proof our observation. As you can see on graph **our theoretical model adequate to reality**. We proof why the **Slinky´s lower part not moved after dropping** (The acceleration of lower part is zero, a=-g, g+(-g)=0m.s^{-1}).

Other conclusions: We work only with plastic slinky. We approximate the air resistance. We compare the equations for a classical spring and for Slinky.

Amplitude: $u(t) = \delta \cos \omega t$ [m]

Velocity: $v = -\delta\omega \sin \omega t$ [m.s^{-1}]

Acceleration: $a = -g \cos \omega t$ [m.s^{-2}]

References:

[1] http://en.wikipedia.org/wiki/Slinky