8. Pebble Skipping

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Problem

It is possible to throw a flat pebble in such a way that it can bounce across a water surface. What conditions must be satisfied for this phenomenon to occur?

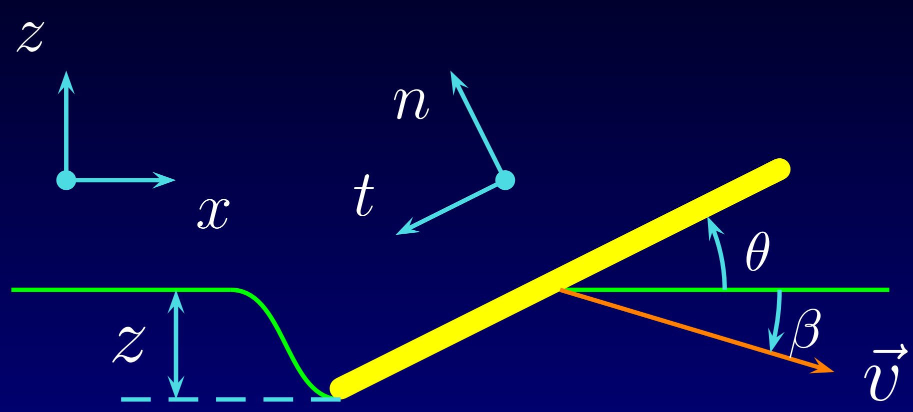
Overview

- Bounce process
  - Experiments
- Rebound conditions
- Computer simulation
- Comparison with experiment

Observation

Video

General Model



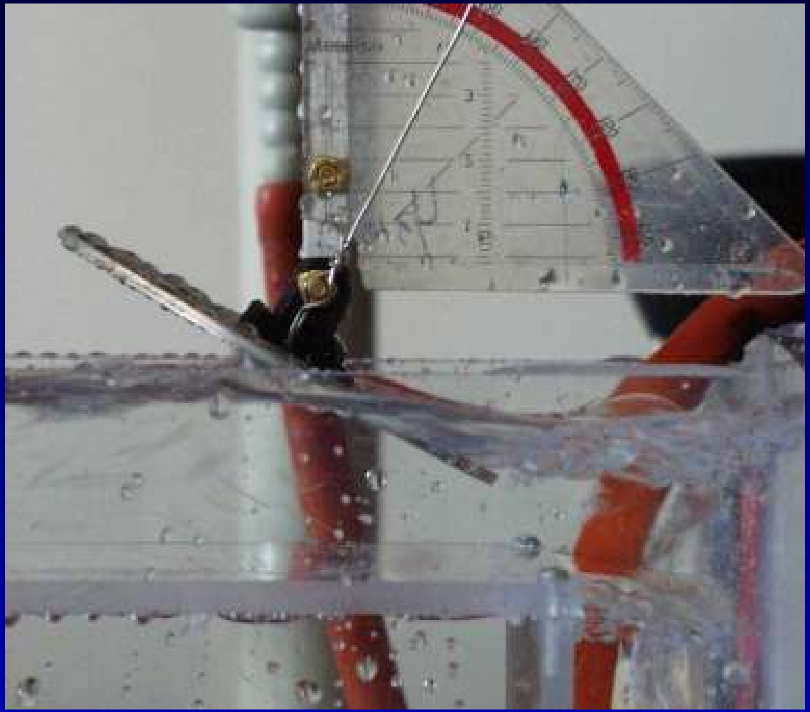
- M - stone mass, $R = a/2$ - radius
- v - stone velocity
- θ - tilt angle, β - incidence angle
- z - depth of the immersed edge
- $S_{\text{im}} = f(z)$ - immersed surface
- Ω - angular velocity

Forces Acting

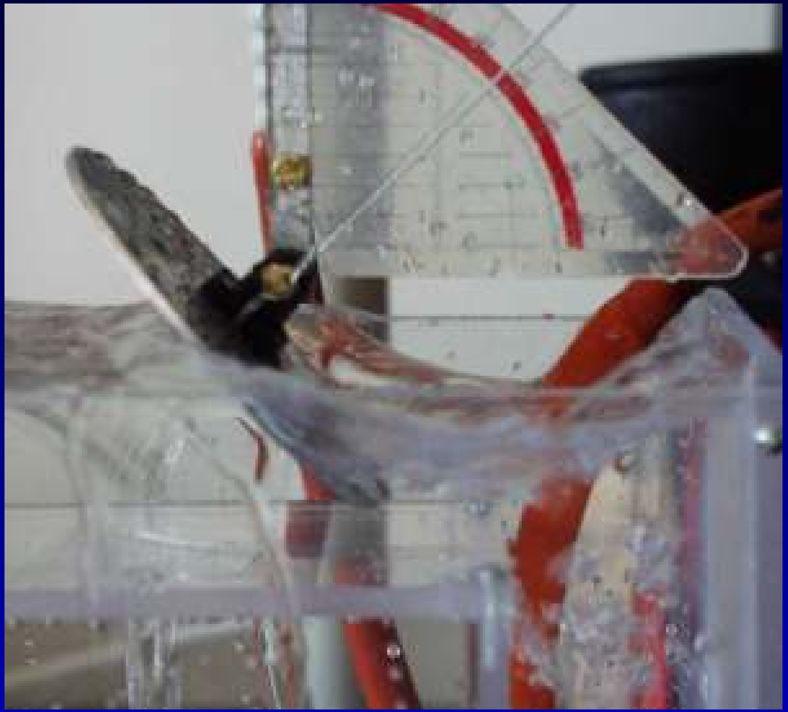
- Gravity
- Hydrodynamic force:
  $\mathcal{R}e \sim 10^6$
  $\mathbf{F} = \mathbf{n}F_n + \mathbf{t}F_t = \mathbf{x}F_x + \mathbf{z}F_z$
  $F_x = \frac{1}{2}C_x \rho v^2 S_{\text{im}};$ $\qquad F_z = \frac{1}{2}C_z \rho v^2 S_{\text{im}}$

Flow for Different Angles

$\theta = 25^\circ$

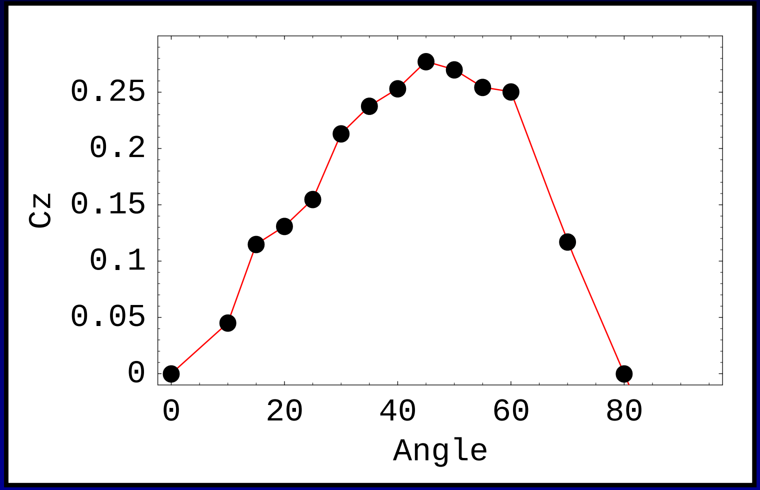


$\theta = 45^\circ$

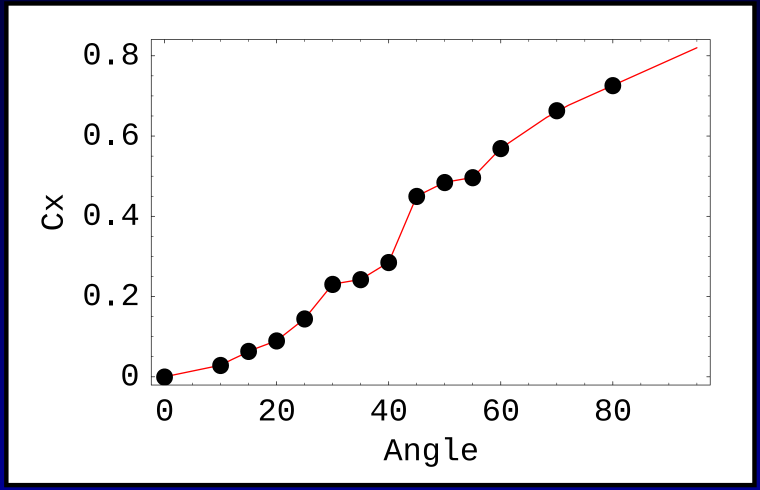


- Stationary flow $\neq$ bounce
- Splashes etc. neglected

Results



$C_z(\theta)$



$C_x(\theta)$

- Linear interpolation
- Basis for further computations

Motion Equations

- Equations of motion:

$$M\ddot{x} = -\frac{1}{2}\rho v^2 C_x(\theta, \beta) S_{\text{im}}(|z|, \beta)$$
$$M\ddot{z} = -Mg + \frac{1}{2}\rho v^2 C_z(\theta, \beta) S_{\text{im}}(|z|, \beta)$$

- $v^2 = \dot{x}^2 + \dot{z}^2$
- $\beta = -\arctan\frac{\dot{z}}{\dot{x}}$
- $S_{\text{im}} = 0$ for $z > 0$

Mass Influence

- Dimensionless mass

$$k = \frac{16M}{\pi \rho a^3}$$

- Physical meaning: quotient of stone mass and mass of water disturbed
- The stone:
  - $M = 50\,\mathrm{g}$
  - $R = 3\,\mathrm{cm}$
  - $k = 1.2$

Angular Stability

- Euler's equations
- Differential equation:

$$\ddot{\theta} - (\theta - \theta_0)\Omega^2 = \frac{\mathcal{M}}{I_1}$$

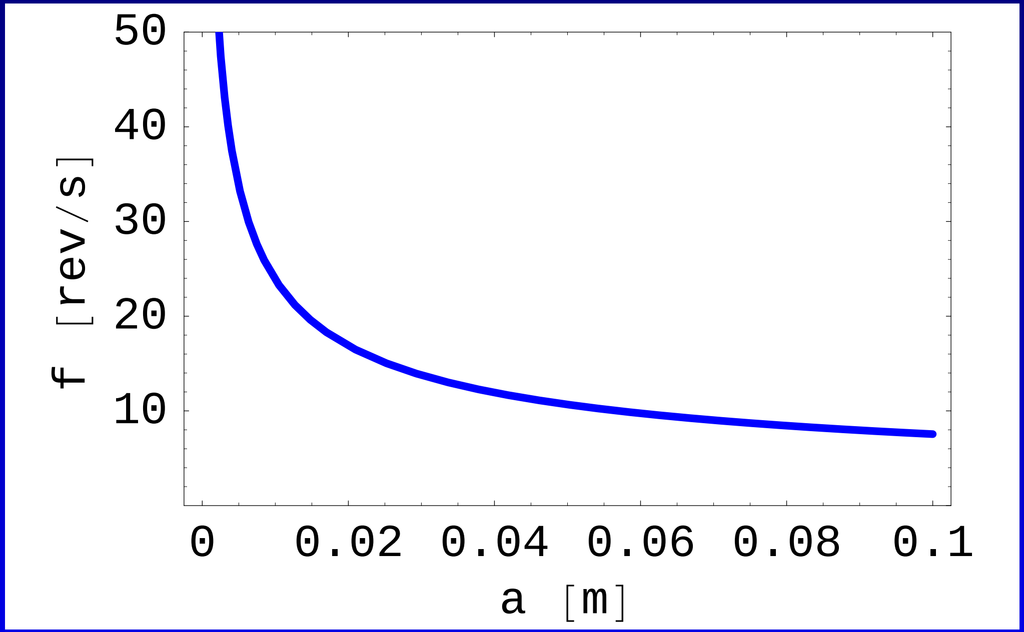
- $\mathcal{M} \sim MgR$
- Condition:

$$\theta > \Delta\theta = \frac{2g}{R\Omega^2}[1 - \cos(\Omega T_0)]$$

Angular Velocity Condition

$$f > \frac{1}{\pi}\sqrt{\frac{g}{\theta a}}$$

- Note:
  - $f_{\text{min}} \propto a^{-\frac{1}{2}}, \quad v_{\text{min}} \propto a^{\frac{1}{2}}$



Restrictions on β

- β must be small
- Jump height:

$$h = \frac{v^2}{2g} \sin^2 \beta$$

- $h \gtrsim |z_{\text{min}}| \approx 1\,\text{cm}$ (“escape” the cavity)
- Condition:

$$\beta > \frac{\sqrt{2g z_{\text{min}}}}{v}$$

- $v \approx 5\,\frac{\text{m}}{\text{s}}$:

$$\beta > 5^\circ$$

Condition I - Immersed Depth

- $\beta \ll 1, \quad v^2 \approx \dot{x}^2 \approx \text{const}, \quad \theta = \text{const}$
- Equation for z can now be solved independently
- Condition: $|z_{\text{min}}| < a / \sin\theta$

Square shape

$$v_0 > v_{\text{im}} = \sqrt{\frac{\pi}{4}} \sqrt{\frac{gak}{C_z}}$$

Circular shape

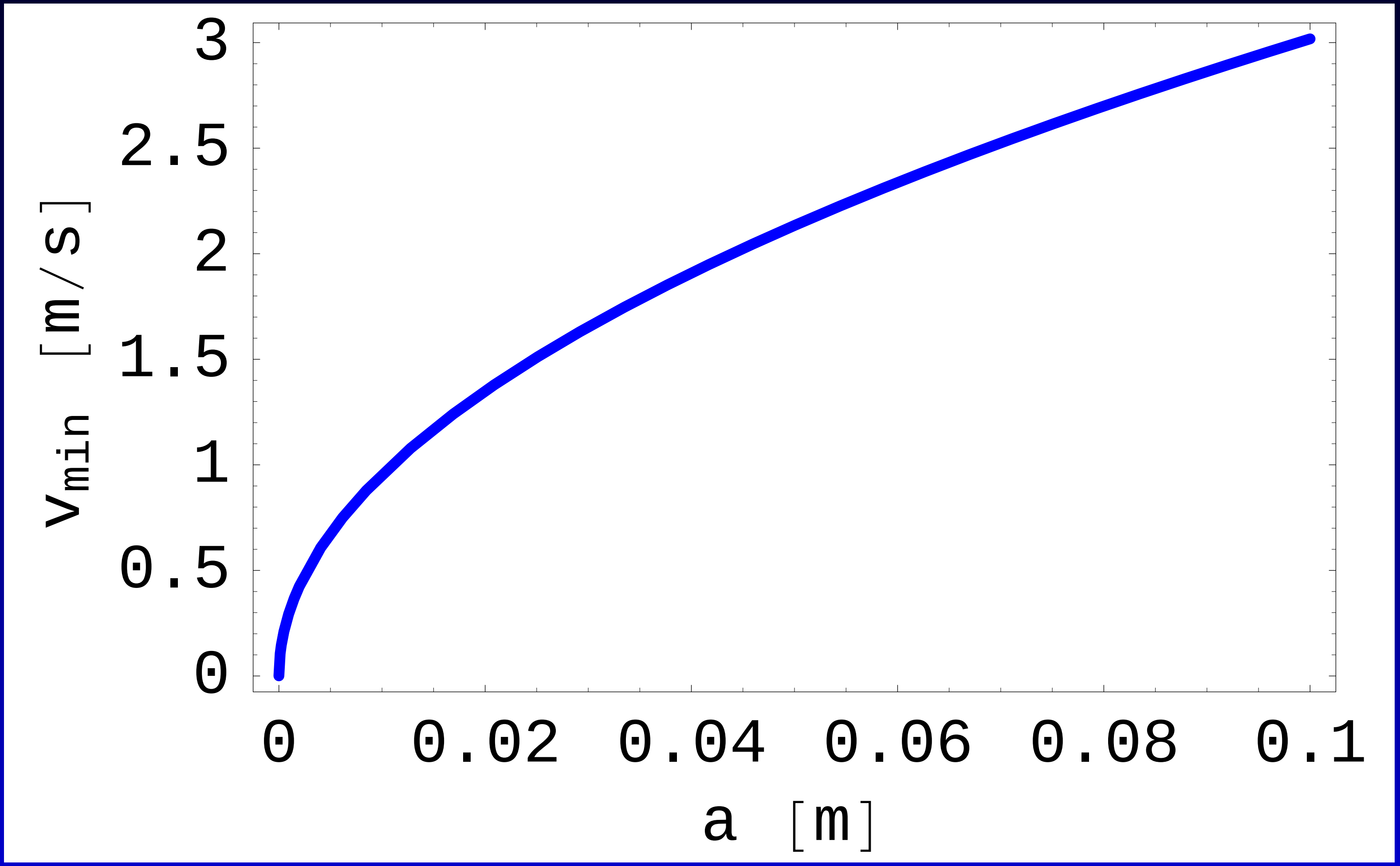
$$v_0 > v_{\text{im}} = \sqrt{\frac{gak}{C_z}}$$

Condition II - Energy Loss

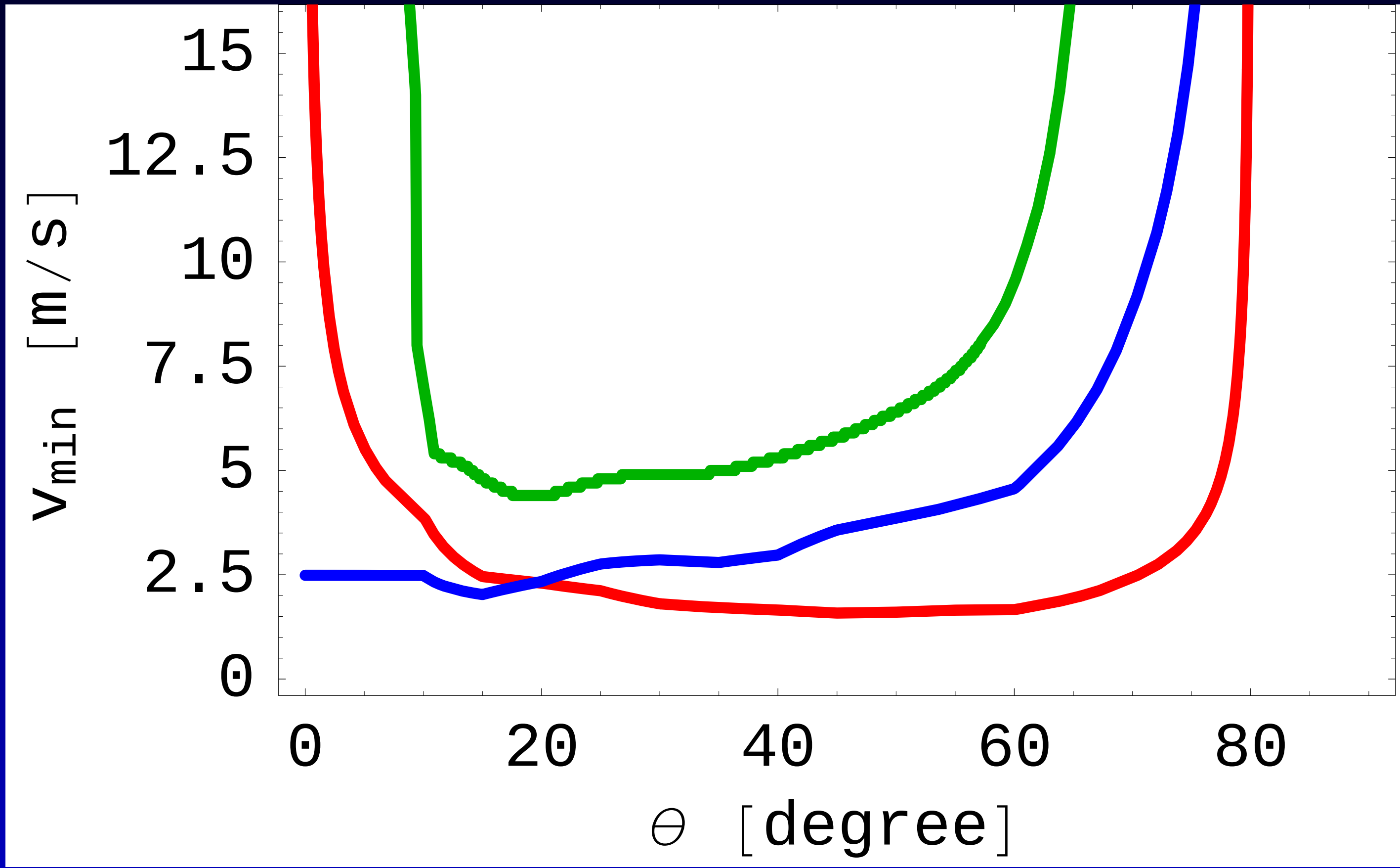
- $v \approx \text{const}$
- During the collision: $F_x = \frac{C_x}{C} F_z, \quad F_z \sim Mg$
- Energy loss: $\Delta W = -F_x l$ with $l = v_{x0} T_{\text{col}}$
- Collision time: $T_{\text{col}} = 5.6 \frac{a}{v_x} \sqrt{\frac{k \sin(\theta)}{C_z}}$
  $(l \neq f(v_x))$
- Condition $\Delta W < \frac{1}{2} M v_{x0}^2$ gives:

$$v_{x0} > 3.3 \sqrt{\frac{C_x}{C_z} g a \sqrt{\frac{k \sin \theta}{C_z}}}$$

Bounce Velocity $v_{\mathbf{min}}(a)$

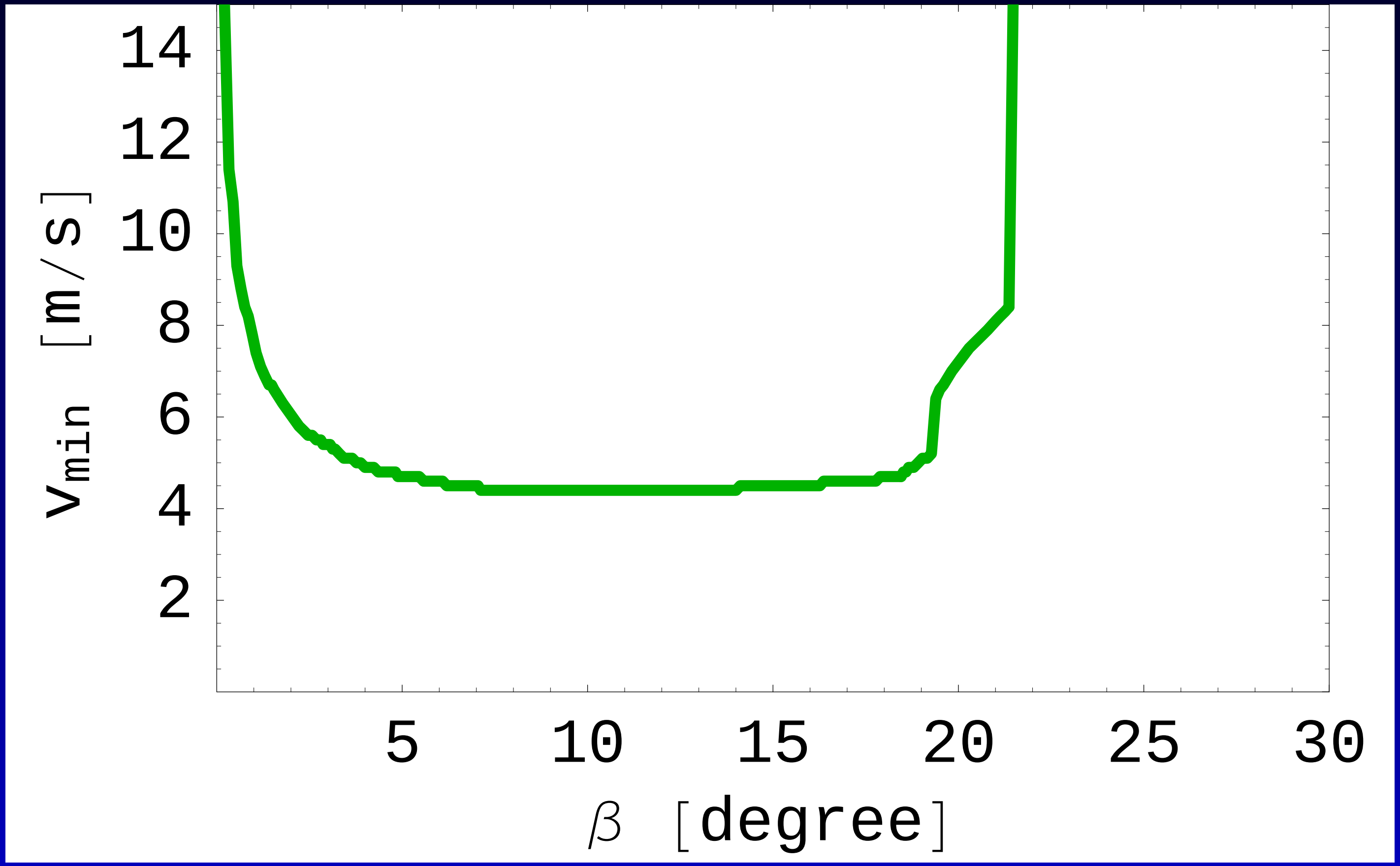


Bounce Velocity $v_{\mathbf{min}}(\theta)$



- Max immerse condition (red)
- Frictional condition (blue)
- Simulation results (green)

Bounce Velocity $v_{\mathbf{min}}(\beta)$



Simulation results

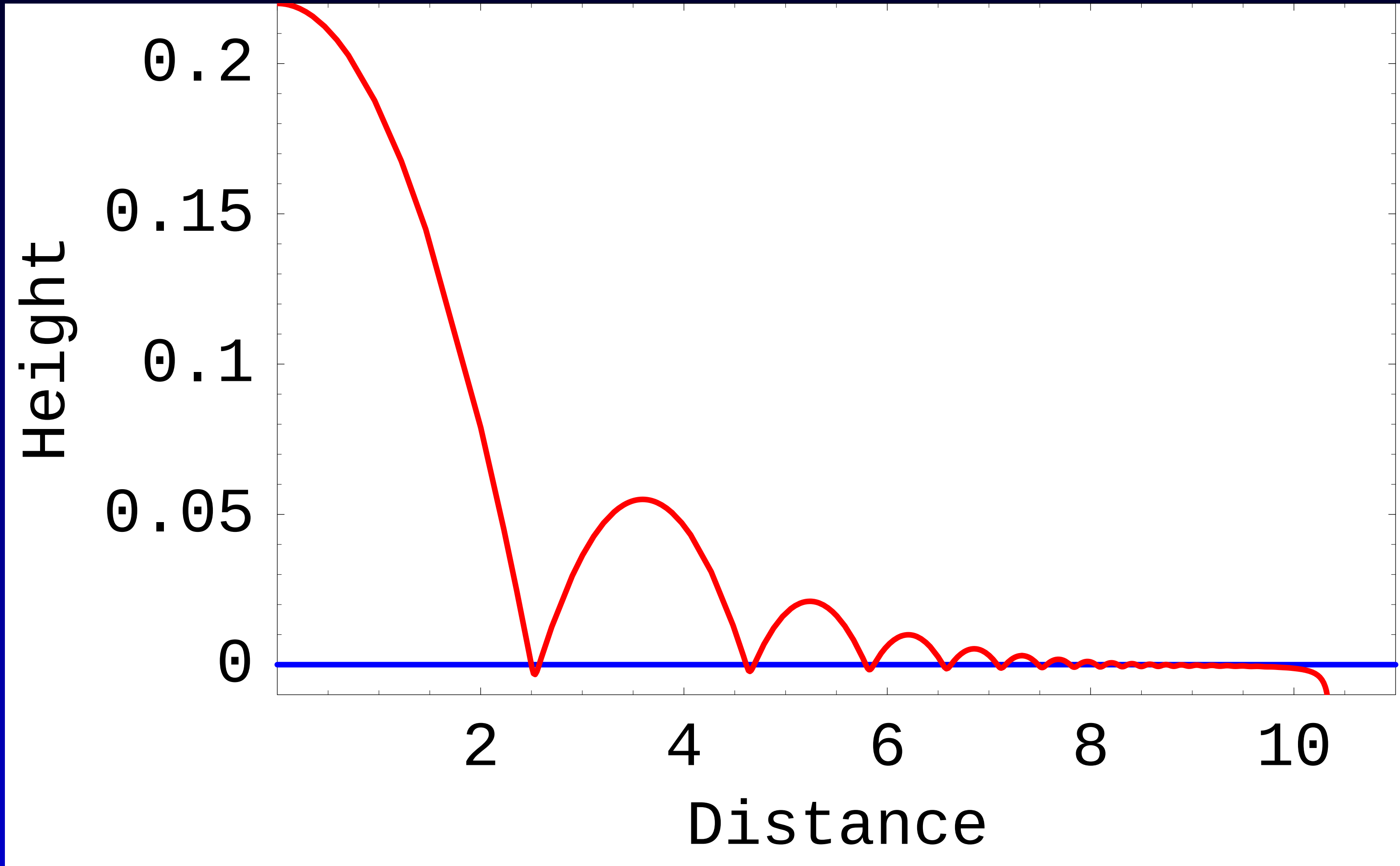
Summary - Conditions

- Spin condition
- Angular condition
- Velocity condition

Multiple Bounces

- Energy loss
- Angular destabilization
- Jumps become shorter ("pitty-pat")
- Last bounce doesn't satisfy the "Conditions"

Multiple Bounces



Simulation results

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