16. Small Fields

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Problem

Construct a device based upon a compass needle and use your device to measure the Earth’s magnetic field.
Overview

- Our Device
- Absolute measurement
  - Method
  - Results / Error analysis
- Relative Measurement
  - Method
  - Results / Error analysis
- Verification of Results
Before we get started...

The Earth’s magnetic Field is relatively small!

Any metallic or electric objects can strongly influence measurement accuracy!

Try to minimize any influences!
Our Device

- Compass needle with angle/B-field-scale
- Additional bar magnet
- Non-magnetic materials
- Measurement of horizontal *and* vertical component possible
Absolute Measurement Method

- Classic method according to Gauß

- Simple devices:
  - Compass needle
  - Bar magnet with dipole moment $m$

- Two measurements:
  - Torsion oscillation of bar magnet: $mB_e$
  - Compass needle deviation: $m / B_e$
Torsion Oscillation

- Moment of inertia of cylindric bar:

\[ J = M \left( \frac{L^2}{12} + \frac{a^2}{4} \right) \]

- Mechanic torque:

\[ T_{\text{mechanic}} = J\ddot{\alpha} \]
Torsion Oscillation

- $\alpha = \angle (\vec{m}, \vec{B}_e)$

- Magnetic torque:
  
  $$T_{\text{magnetic}} = mB_e \sin \alpha \approx mB_e \alpha$$

- Mechanic torque due to thread negligible:
  
  $$\frac{T_{\text{thread}}}{T_{\text{magnetic}}} \ll 1\%$$
Torsion Oscillation

- Differential equation for harmonic oscillation:

\[ T_{\text{mechanic}} = -T_{\text{magnetic}} \]
\[ J\ddot{\alpha} = -mB_e\alpha \]
\[ \ddot{\alpha} + \frac{mB_e}{J}\alpha = 0 \]

\[ \implies \frac{mB_e}{J} = \omega^2 = \left(\frac{2\pi}{T}\right)^2 \]
Torsion Oscillation Result

- **End formula:**

\[ mB_e = \frac{4\pi^2}{T^2} M \left( \frac{L^2}{12} + \frac{a^2}{4} \right) \]

- **Measurement:**

\[ L = 199 \pm 1 \text{ mm}; \quad 2a = 10 \pm 0.1 \text{ mm} \]

\[ M = 101 \pm 0.5 \text{ g}; \quad 50T = 393 \pm 0.5 \text{ s} \]

\[ mB_e = (2.13 \pm 0.02) \cdot 10^{-4} \text{ Am}^2 \text{ T} \]

\[ \varepsilon(mB_e) = 1.1\% \]
Needle Deviation

- \( \tan \varphi = \frac{B_m}{B_e} \) true for \( r \gg \frac{l}{2}, \frac{L}{2} \)

- With permanent magnet:

\[
\tan \varphi = \frac{1}{B_e} \frac{\mu_0 m}{2\pi r^3} \left( 1 - \frac{\zeta}{r^2} \right)
\]
**Needle Deviation**

\[
\frac{m}{B_e} = \tan \varphi \frac{2\pi r^3}{\mu_0} \left( \frac{r^2}{r^2 - \zeta} \right)
\]

- Eliminating \( \zeta \) by measuring two different \( r \)
- End Formula:

\[
\frac{m}{B_e} = \frac{2\pi}{\mu_0} \left( \frac{r_2^5 \tan \varphi_2 - r_1^5 \tan \varphi_1}{r_2^2 - r_1^2} \right)
\]
Needle Deviation Result

• Measurements:

\[ r_1 = 80 \pm 0.1 \text{ cm}; \quad \varphi_1 = 11.5^\circ \pm 0.5^\circ \]
\[ r_2 = 60 \pm 0.1 \text{ cm}; \quad \varphi_2 = 27.5^\circ \pm 0.5^\circ \]

• Result:

\[ \frac{m}{B_e} = (4.66 \pm 0.56) \cdot 10^5 \text{ Am}^2\text{T}^{-1} \]

\[ \varepsilon \left( \frac{m}{B_e} \right) = 12.0\% \]
Absolute Magnetic Field Value

- Merging both measurements:

\[ B_e = \sqrt{\frac{mB_e}{m/B_e}} \]

\[ \varepsilon(B_e) = \frac{1}{2} \left( \sqrt{\varepsilon(mB_e)^2 + \varepsilon \left(\frac{m}{B_e}\right)^2} \right) \]

- Result:

\[ B_e = (2.14 \pm 0.13) \cdot 10^{-5} \text{ T} \]
\[ \varepsilon(B_e) = 6.0\% \]
Relative Measurement Method

- For $r_0 = 49$ cm we get $\varphi_0 = 45.0^\circ$
- $B_e = B_m = 21.4 \mu T$ true for $\varphi_0 = 45.0^\circ$

$$\tan \varphi = \frac{B_m}{B_e} \Rightarrow B_e = \frac{21.4}{\tan \varphi} \mu T$$
Relative Measurement Method

- Error propagation for relative measurement:

\[ \varepsilon(B_{e,\text{rel}}) = \sqrt{\varepsilon(B_{e,\text{abs}})^2 + \varepsilon(\tan \varphi_0)^2 + \varepsilon(\tan \varphi)^2} \]

- According to error propagation rules:

\[ \varepsilon(\tan \varphi_i) = \frac{1 + \tan^2 \varphi_i}{\tan \varphi_i} \Delta \varphi_i; \quad \Delta \varphi_i = \frac{0.5^\circ}{180^\circ \pi} \]

- Minimum error for \( \varphi \approx \varphi_0 \approx 45^\circ \):

\[ \varepsilon(B_{e,\text{rel}}) = 6.5\% \]
Measuring Vertical Component

- Measured angle for vertical intensity $26.5^\circ$:

$$B_{e, \text{vertical}} = \frac{21.4}{\tan \varphi_{e, \text{vertical}}} \mu T = 42.9 \mu T$$

- Total intensity:

$$B_{e, \text{total}} = \sqrt{B_{e, \text{horizontal}}^2 + B_{e, \text{vertical}}^2} = 47.9 \mu T$$
Using a helmholtz coil:

\[ B = \frac{\mu_0 n I}{r} \left( \frac{4}{5} \right)^{3/2} = 21.2 \mu T \]
Verification with Literature

Measured intensities compared to data of the US National Geophysical Data Center:

<table>
<thead>
<tr>
<th></th>
<th>Own</th>
<th>NGDC</th>
</tr>
</thead>
<tbody>
<tr>
<td>Horiz</td>
<td>21.4 $\mu$T</td>
<td>21.0 $\mu$T</td>
</tr>
<tr>
<td>Vert</td>
<td>42.9 $\mu$T</td>
<td>42.5 $\mu$T</td>
</tr>
<tr>
<td>Total</td>
<td>47.9 $\mu$T</td>
<td>47.4 $\mu$T</td>
</tr>
</tbody>
</table>
Calibration

- Remember $B_e = \frac{21.4}{\tan \varphi} \mu T$ true for $r = 49$ cm
Summary

- Magnetic field measurement in seconds
- Measurement of horizontal \textit{and} vertical component
- No complicated devices (lasers, coils, etc.)
- Relative accuracy 6.5\% 
- Accuracy can hardly be increased with used compass needle
Summary

- To avoid influences due to environment:
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