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- **Brazilian team**

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Problem 8 - Pebble Skipping

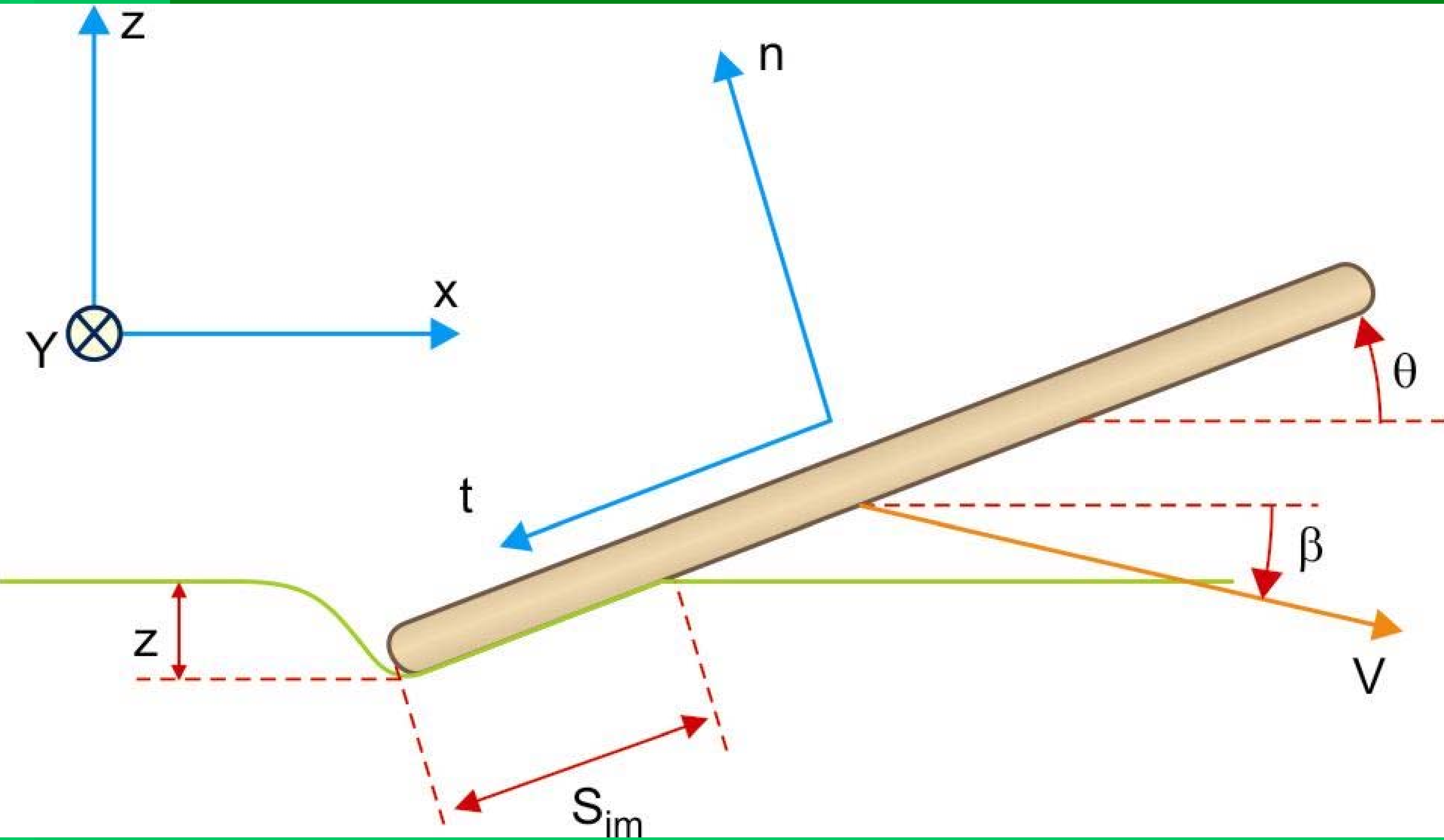
“It is possible to throw a flat pebble in such way that it can bounce across a water surface. What conditions must be satisfied for this phenomenon to occur?”

Introduction - Curiosities

- World Record: J. Coleman-McGhee in 1992, 38 rebounds
- Lyderic Bocquet

Initial considerations

- Mass: M
- Thin stone
- Flat stone
- Flat surface
- The velocity V is assumed to lie in a symmetric plane of the stone


z
n
x
Y
θ
t
β
z
V
S_im

Flow

- Reyndds' number:
- $Re = Va/\nu$
- ν= kinect viscosity ~10^-6 for the water
- Re~10^5 >> 1
- Flow of wet water

The force in a turbulent flow is given by:

$$\mathbf{F} = \frac{1}{2} C_l \rho_\omega V^2 S_{im} \mathbf{n} + \frac{1}{2} C_f \rho_\omega V^2 S_{im} \mathbf{t},$$

Where C(l) is the lift coefficient,

C(f) is the friction coefficient,

ρ(w) is the mass density of water,

S(im) is the area of the immersed surface,

Equations of motion - Considerations

- Collision time: the stone is partially immersed in water.
- Incidence angle θ is constant during the collisional process.
- The origin of time is the instant when the edge of the stone reaches the water surface.

- During the collisional process, the equations of motion for the center of mass velocity are:

$$M \frac{dV_x}{dt} = -\frac{1}{2}\rho_\omega V^2 S_{\text{im}} (C_l \sin\theta + C_f \cos\theta),$$

$$M \frac{dV_z}{dt} = -Mg + \frac{1}{2}\rho_\omega V^2 S_{\text{im}} (C_l \cos\theta - C_f \sin\theta),$$

g= gravity aceleration

$$V^2 = V_x^2 + V_z^2 \implies V^2 \simeq V_{x0}^2 + V_{z0}^2 \simeq V_{x0}^2$$

$$C = C_l \cos\theta - C_f \sin\theta \simeq C_l$$

Square Stone

- $$S_{im} = a\,|z|\,/\,sin\,\theta$$

with a the length of one edge of the stone.

- The equation for z becomes:

$$M \frac{d^2 z}{dt^2} = -Mg - \frac{1}{2} \rho_w V_{x0}^2 C \frac{az}{\sin\theta}$$

- The characteristic frequency is given by:

$$\omega_0^2 = \frac{C \rho_\omega V_{x0}^2 a}{2M \sin\theta}$$

- Rewriting equation for z, we have:

$$\frac{d^2 z}{dt^2} + \omega_0^2 z = -g$$

- $$z(t) = -\frac{g}{\omega_0^2} + \frac{g}{\omega_0^2}\cos \omega_0 t + \frac{V_{z0}}{\omega_0}\sin \omega_0 t$$

- Such equation characterizes the collisional process of the stone with water. The maximal depth attained by the stone during collision is

$$|z_{max}| = \frac{g}{\omega_0^2}\left[1 + \sqrt{1 + \left(\frac{\omega_0 V_{z0}}{g}\right)^2}\,\right]$$

- The pebble can't be totally immersed to rebound, so, the rebound condition is:

$$|z_{max}| < a \sin \theta$$

- Critical velocity (minimum to the stone rebound:

$$V_{x0} > V_c = \frac{\sqrt{\dfrac{4Mg}{C\rho_w a^2}}}{\sqrt{1 - \dfrac{2\tan^2\beta M}{a^3 C \rho_w \sin\theta}}}$$

Where the incidence angle β is defined as $V_{z0} / V_{x0} = \tan \beta$

Circular Stone

- $S_{im}(s) = R^2 \left[\arccos(1 - s/R) - (1 - s/R)\sqrt{1 - (1 - s/R)^2}\right]$
- with $s = |z| / \sin\theta$
- $$V_{x0} > V_c = \frac{\sqrt{\dfrac{16\,M\,g}{\pi\,C\rho_\omega\,a^2}}}{\sqrt{1 - \dfrac{8\,M\,\tan^2\beta}{\pi\,a^3 C\rho_\omega \sin\theta}}}$$

Energy dissipation

- The friction force is a non-conservative force. The mechanism of energy dissipation leads to another minimum velocity condition. The decrease in the kinetic energy in the x direction is given by:

$$W \equiv \frac{1}{2} M V_{xf}^2 - \frac{1}{2} M V_{x0}^2 = -\int_0^{t_{coll}} F_x(t)\, V_x(t)\, dt$$

- Where, t_{coll} is the collision time
- $F_x = \frac{1}{2}\tilde{C}\rho_\omega V_x^2 S_{\text{im}}$ is the x-component of the reaction force
- With $\tilde{C} = C_l \sin\theta + C_f \cos\theta$

  $C = C_l \sin\theta - C_f \cos\theta$

- $$\int_0^{t_{coll}} F_x(t)\, V_x(t)\, dt \simeq V_{x0} \int_0^{t_{coll}} F_x(t)\, dt$$

- $$F_x(t) = \mu\, F_z(t)$$ with $$\mu = \tilde{C}/C$$

- $$\langle F_z(t) \rangle = t_{coll}^{-1} \int_0^{t_{coll}} F_z(t)\, dt$$

- $$\langle F_x(t) \rangle \simeq \mu\, M\, g$$

- $t_{\text{coll}} \sim 2\pi / \omega_0$
- Loss in kinetic energy:

$$W \simeq -\mu M g V_{x0} \frac{2\pi}{\omega_0} = -\mu M g \ell$$

- $$\ell = V_{x0} \frac{2\pi}{\omega_0} = 2\pi \sqrt{\frac{2 M \sin\theta}{C \rho_w a}}$$
- $$\omega_0^2 = \frac{C \rho_w V_{x0}^2 a}{2 M \sin\theta}$$

- $$\frac{1}{2} M V_{x0}^2 > |W| = \mu M g \ell$$
- $$V_{x0} > V_c = \sqrt{2 \mu g \ell}$$

Spin

- It is obvious that the rebound of the stone is optimized when θ is small and positive. If after a collision the stone is put in rotation around the y-axis, that is, $d\theta/dt$ is not equal to zero, its orientation would change during free flight, and the incidence angle θ for the next collision has little chance to still be in a favorable situation. So, a stone spin is needed, for its angular motion stability.

- **Torque**
- $$\mathcal{M}_{lift} = OP \, . \, F_{\mathrm{lift}} e_y$$

- O is the center of mass of the stone
- P is the center of mass of the immersed area

- Stabilizing torque: **Giroscopic Effect**

- $$I_1 \frac{d\omega_1}{dt} - \omega_2\omega_3(I_2 - I_3) = N_1$$
$$I_2 \frac{d\omega_2}{dt} - \omega_1\omega_3(I_3 - I_1) = N_2$$
$$I_3 \frac{d\omega_3}{dt} - \omega_1\omega_2(I_1 - I_2) = N_3$$
- $\omega_1 = \dot{\theta}$
- $I_1 = I_3 \equiv J_1$
- $I_2 \equiv J_0$
- $N_1 \equiv \mathcal{M}_\theta$
- $N_2 = N_3 = 0$

- $$\omega_3 = \frac{J_1 - J_0}{J_1}\,\dot{\phi}_0\,(\theta - \theta_0)$$
- $$\ddot{\theta} + \omega^2\,(\theta - \theta_0) = \frac{M_\theta}{J_1}$$
- $$\omega = [\,(J_0 - J_1)\,/\,J_1\,]\,\dot{\phi}_0$$ if $$\dot{\phi}_0 = 0$$
- $$\delta\theta = [\,\theta - \theta_0\,]_{max} \ll 1$$

- $(J_0 - J_1)/J_1 \sim 1$ and $J_1 \sim M R^2$
- $\langle F_Z(t) \rangle \simeq M g$
- $\delta\theta \sim g/(R\omega^2)$
- $OP \sim R$
- $\mathcal{M}_\theta \sim M g R$
- $\delta\theta \ll 1$

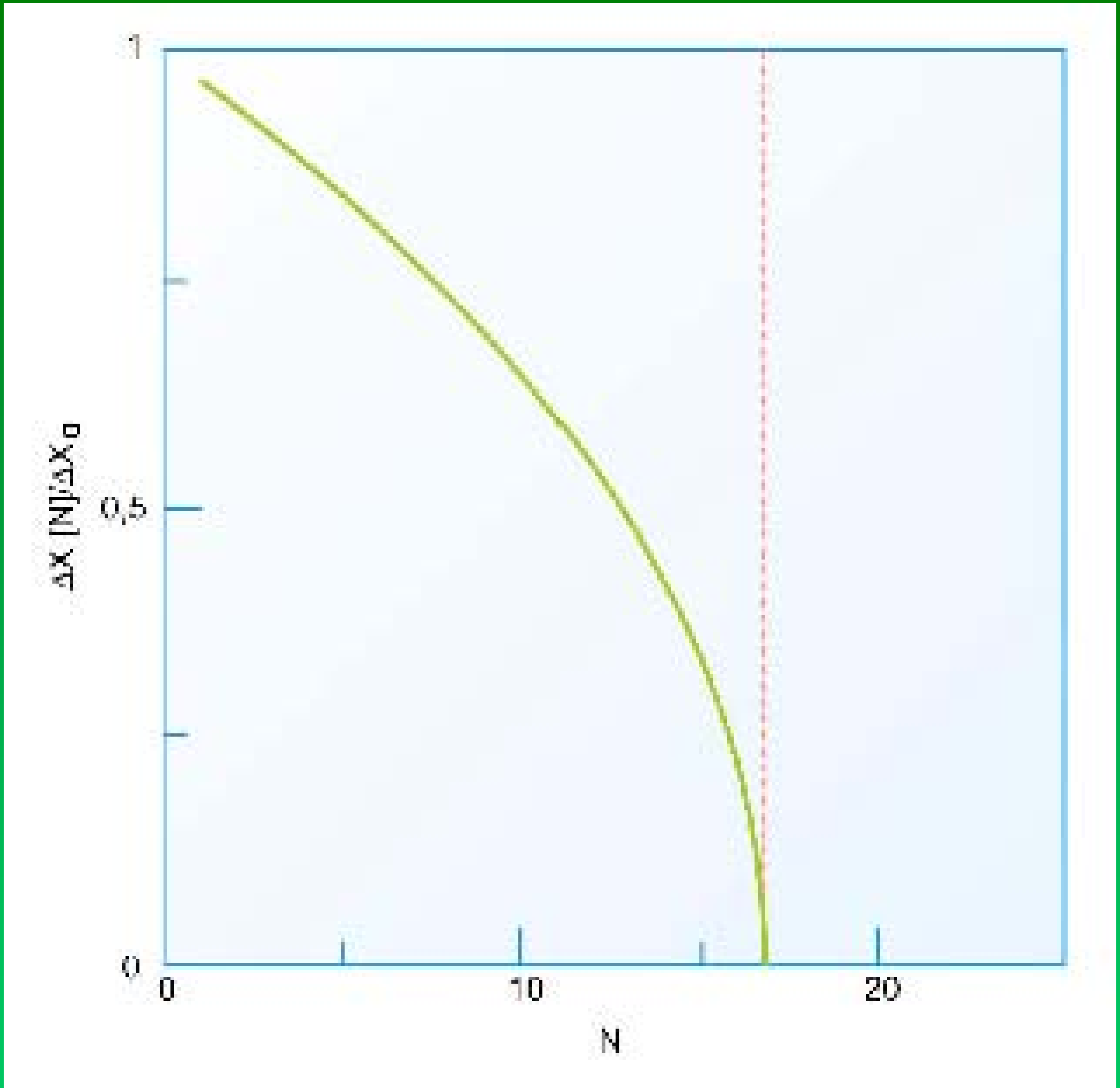
$$\dot{\phi}_0 \sim \omega \gg \sqrt{\frac{g}{R}}$$

Number of bounces

- $$\frac{1}{2} M V_x^2 [N] - \frac{1}{2} M V_x^2 [0] = - N \mu M g \ell$$
- $$V_x^2 [N_c] = 0$$
- $$N_c = \frac{V_x^2 [0]}{2 g \mu \ell}$$

Distance between two sucessive collisions

- $X(t) = V_x t$
- $Z(t) = -\frac{1}{2} g t^2 + |V_z| t$
- $\Delta X = 2\, V_x\, |V_z| \,/\, g$
- $\Delta X[N] = \Delta X_0 \sqrt{1 - \frac{N}{N_c}}$
- $\Delta X_0 = 2\, V_{x0}\, /V_{z0}/ \,/\, g$



- Plot of the (normalized) distance between two successive collisions $\Delta X[N]/\Delta X_0$ as a function of the number of bounces N. The initial velocity is V_{xo}=8m/s, corresponding to N_c=17.

Angular destabilization

- $\delta\theta \sim g/(R\omega^2)$
- $\omega \sim \dot{\phi}_0$
- $\Delta_N \theta \sim N\delta\theta$
- $\Delta_{N_c} \theta \sim 1$
- $N_c \sim \dfrac{R\dot{\phi}_0^2}{g}$

Experience I

Experience II

Animation

Calculation I

- Critical velocity:
- $M = 0.1\ kg,$
- $L = 0.1\ m$
- $C_l \approx C_f \approx 1.$
- $\rho_w = 1000\ kg\ m^{-3}$
- $\beta \sim \theta \sim 20°$
- $V_c \simeq 0.66\ m\ s^{-1}$

$$V_{x0} > V_c = \frac{\sqrt{\dfrac{4M\,g}{C\,\rho_w a^2}}}{\sqrt{1 - \dfrac{2\tan^2\beta\, M}{a^3\, C\,\rho_w \sin\theta}}}$$

Calculation II

- Critical velocity:
- $$V_{x0} > V_c = \sqrt{2\,\mu\, g\,\ell}$$

$$\mu = \tilde{C}/C$$

$$\ell = V_{x0}\,\frac{2\pi}{\omega_0} = 2\pi\sqrt{\frac{2\,M\sin\theta}{C\,\rho_w a}}$$

$\mu = 2.14$

- l = 16.5 cm
- Vc ~ 2,66 m/s

Calculation III

- $$\dot{\phi}_0 \sim \omega \gg \sqrt{\frac{g}{R}}$$
- R = 5 cm
- g = 10 m/s²
- ω >> 14.14 s^-1

Calculation IV

- $$N_c = \frac{V_x^2[0]}{2\, g\, \mu\, \ell}$$
- $N_c = 2 \rightarrow V_x^2[0] = 3.8$ m/s
- $N_c = 3 \rightarrow V_x^2[0] = 4.6$ m/s
- $N_c = 38 \rightarrow V_x^2[0] = 16.4$ m/s

Conclusion

The conditions that must be satisfied to occur a pebble bouncing are:

- An initial velocity in the order of meter per second;
- An initial spin, for pebble's angular stability;
- A low angle (about 20º) with water surface of the stone;
- Pebble's shape: flat and rather circular;