Problem 8.

Pebble skipping

Problem

It is possible to throw a flat pebble in such a way that it can bounce across a water surface. What conditions must be satisfied for this phenomenon to occur?

Basic idea

- The conditions needed for a flat pebble to skip on a water surface are:
  - Initial velocity should be greater than 3 m/s
  - Angle between water surface and the main plane of the pebble (angle of attack) should be between 10° and 30°
  - The pebble has to rotate

Experimental approach

- Parameters influencing the motion of the pebble on water:
  - Pebble characteristics (mass, shape, dimensions)
  - Angle of attack
  - Velocity
  - Rotational velocity

The experiment was divided in two parts:

1. Throwing pebbles on a water surface (lake)
2. Laboratory measurements

1. Throwing real pebbles

- Goals:
    - Determine the optimal shape, size and mass of a skipping pebble
    - Find the best way of throwing skipping pebbles

1. Varying the shape and mass of the pebble

Mass

- A massive pebble needs greater velocity to skip

Shape

- A flat pebble (big contact surface) will skip best

Conclusion

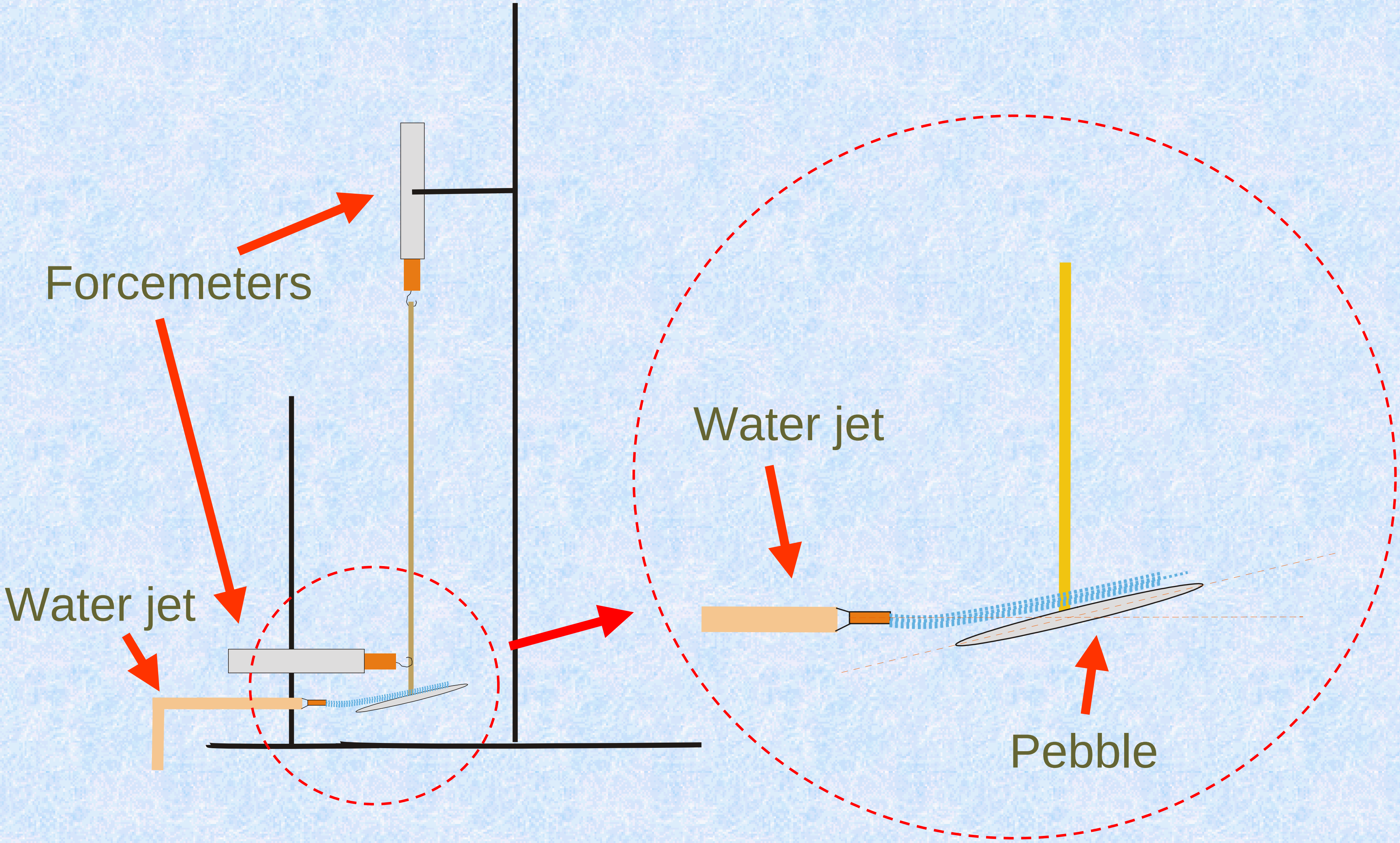
- An ideal skipping pebble should be:
  - Flat
  - Realtively heavy
  - With big surface area
- The shape isn’t as important; most pebbles found in nature are irregular
- Many different, nonideal pebbles will skip too if given an initial velocity large enough

2. Laboratory measurements

What to measure?

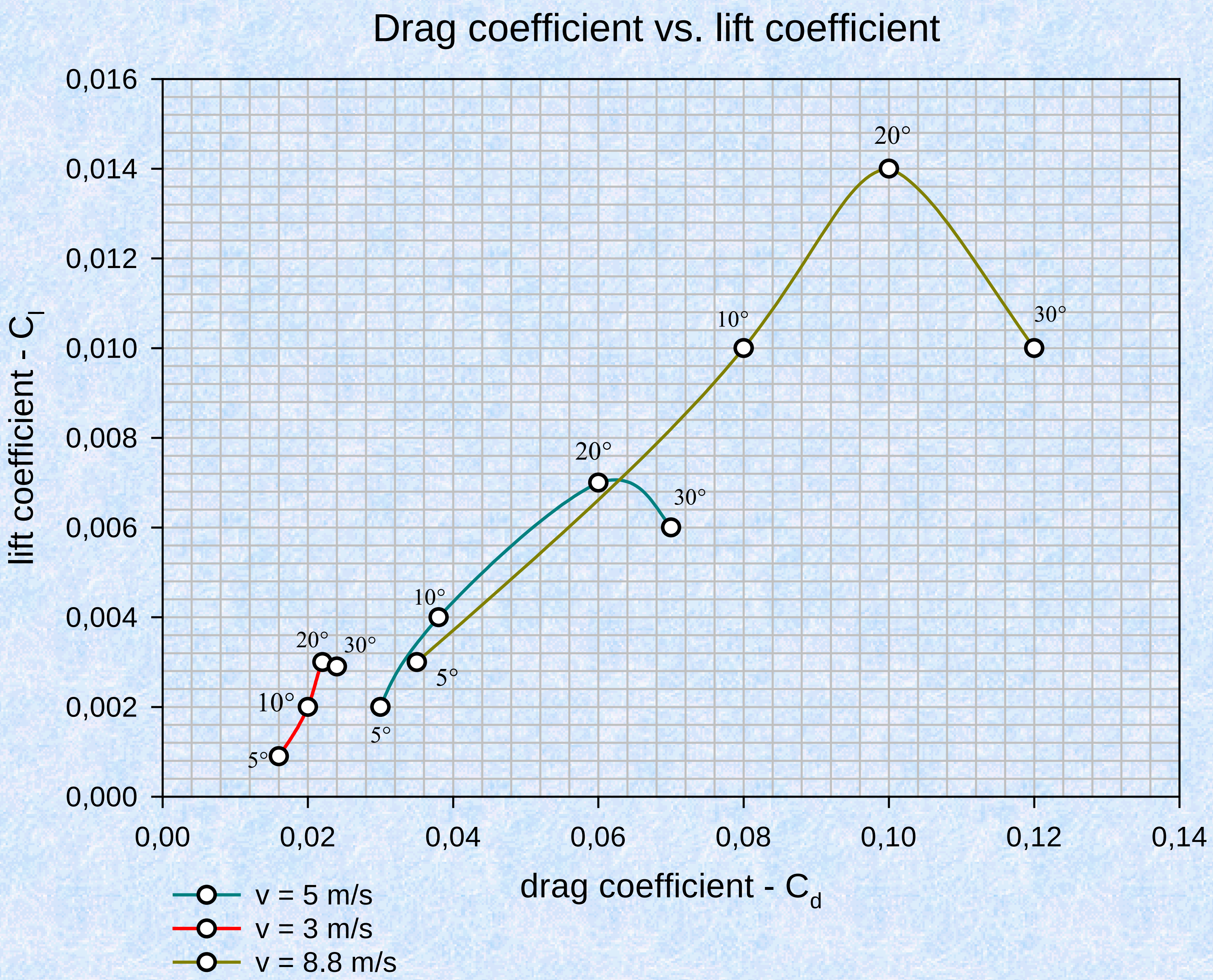
- Lift and drag coefficients with varying
  - Angle of attack
  - Pebble velocity
- Net hydrodinamical force on pebble
- Minimal velocity needed for bouncing

Experimental setup

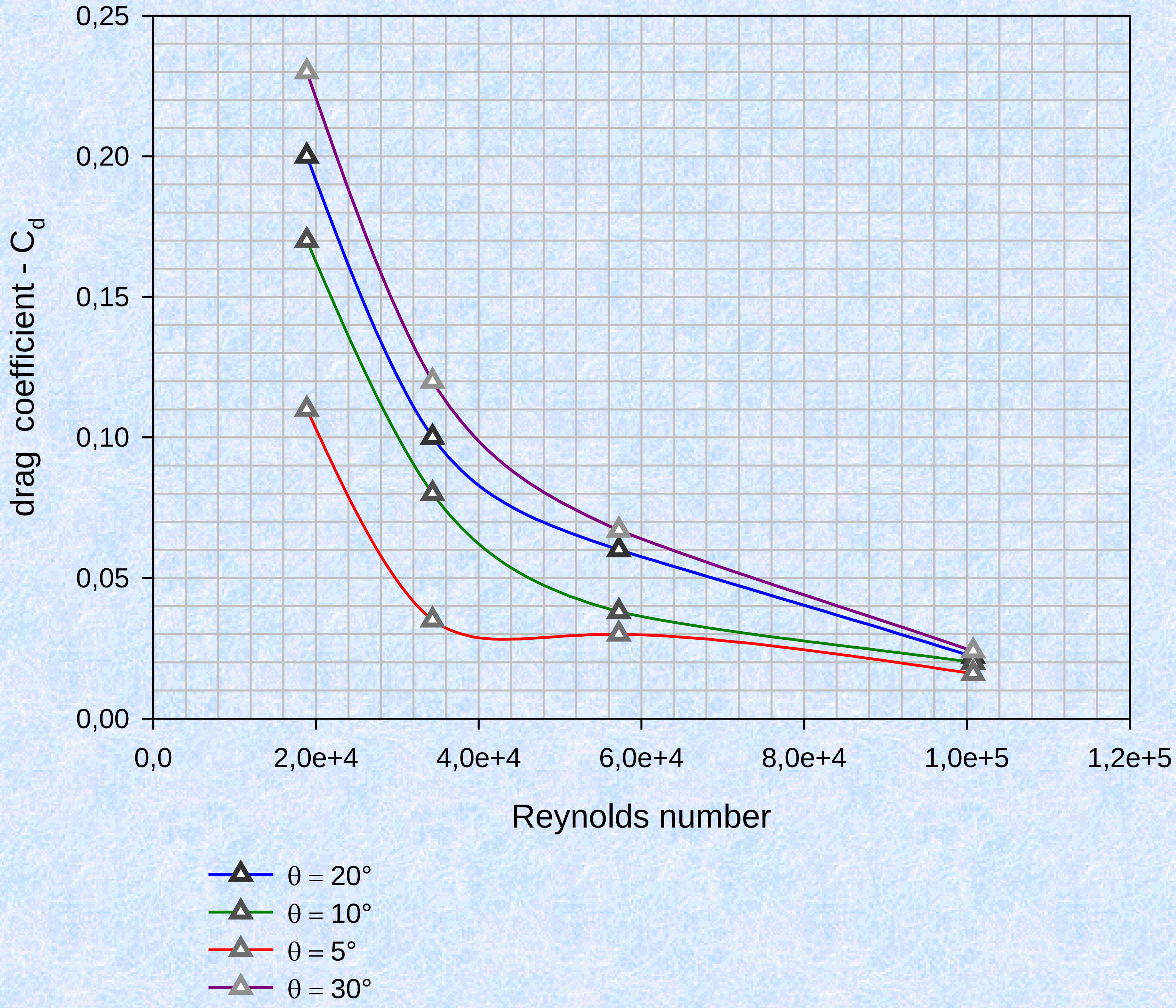


- The measurements had been performed with an idealized pebble model

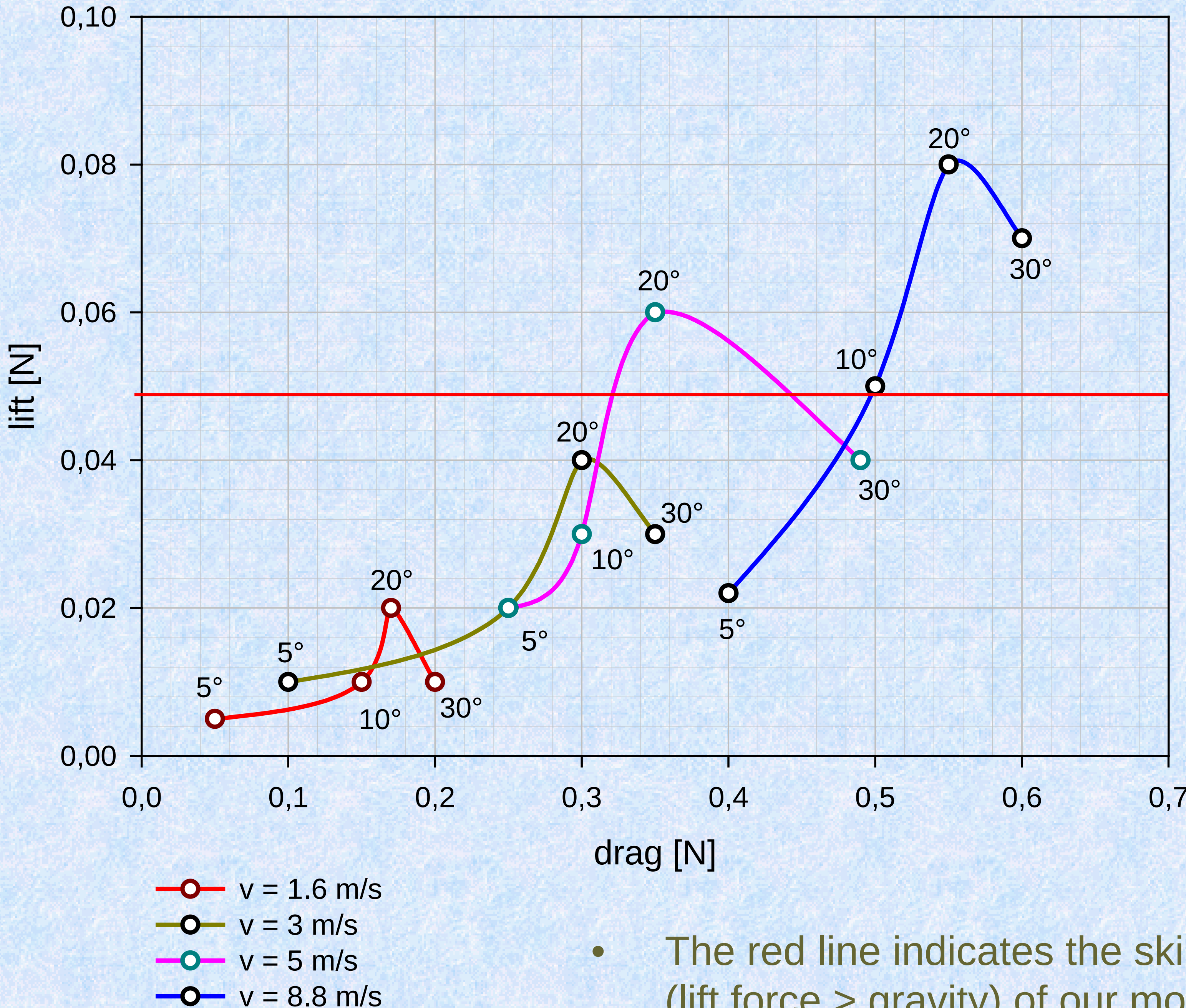
Results



Drag coefficient vs. lift coefficient
lift coefficient - C_l
drag coefficient - C_d
0,016
0,014
0,012
0,010
0,008
0,006
0,004
0,002
0,000
0,00
0,02
0,04
0,06
0,08
0,10
0,12
0,14
20°
10°
30°
20°
30°
10°
5°
20°
30°
10°
5°
5°
v = 5 m/s
v = 3 m/s
v = 8.8 m/s



0,25
0,20
0,15
0,10
0,05
0,00
drag coefficient - C_d
0,0
2,0e+4
4,0e+4
6,0e+4
8,0e+4
1,0e+5
1,2e+5
Reynolds number
θ = 20°
θ = 10°
θ = 5°
θ = 30°



- The red line indicates the skip limit (lift force > gravity) of our model

Conclusion

- Angle of attack
  - For our model the optimal throwing angle is about 20°
  - The minimal throwing angle for pebble velocity 8.8 m/s is 10°
- Minimal velocity
  - The jump limit of our model was at about 3.8 m/s for optimal angle of attack
  - For other angles the minimal velocity is greater

Theoretical approach

Forces acting on the pebble during contact

- Hydrodinamical forces:

Lift

$$F_l = \frac{1}{2} C_l S_{im} \rho_w v^2$$

Drag

$$F_d = \frac{1}{2} C_d S_{im} \rho_w v^2$$

Gravity

$$F_g = mg$$

C_l – lift coefficient

C_d – drag coefficient

ρ_w – density of water

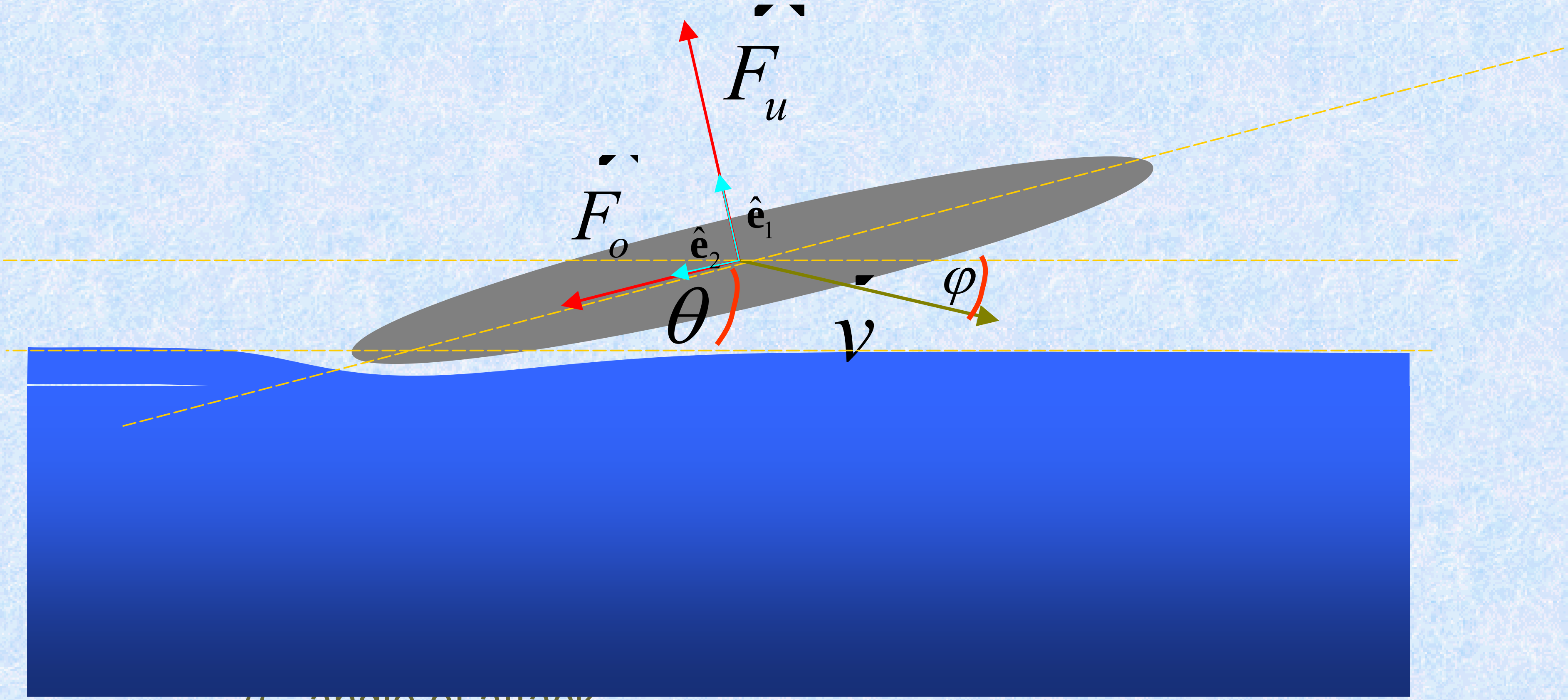
v – pebble velocity

S_{im} – immerged surface of pebble

m – pebble mass

g – free fall acceleration

Defining the coordinate system



θ – angle of attack

v – pebble velocity

$\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2$ - unit vectors

φ – angle between surface and velocity vector

Equation of motion

- In components:

$$m\frac{dv_x}{dt} = -\frac{1}{2}\rho_t v^2 S_u \left(C_l \sin\vartheta + C_d \cos\vartheta\right)$$

$$m\frac{dv_z}{dt} = -mg + \frac{1}{2}\rho_w v^2 S_{im} \left(C_d \cos\vartheta - C_l \sin\vartheta\right)$$

v_x – x – component of velocity

v_z – z – component of velocity

θ - angle of attack

Simplifying the equation of motion

$$v^2 \approx v_{x0}^2 + v_{z0}^2 \approx v_{x0}^2$$

v_{x0} – *x* – component of velocity

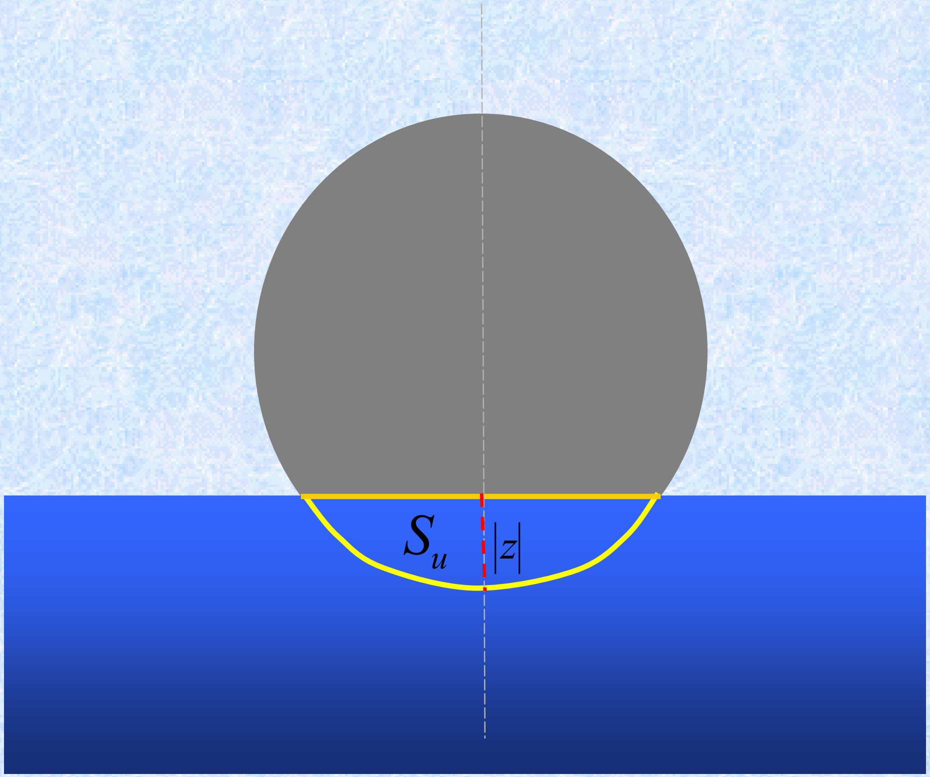
v_{z0} – *z* – component of velocity

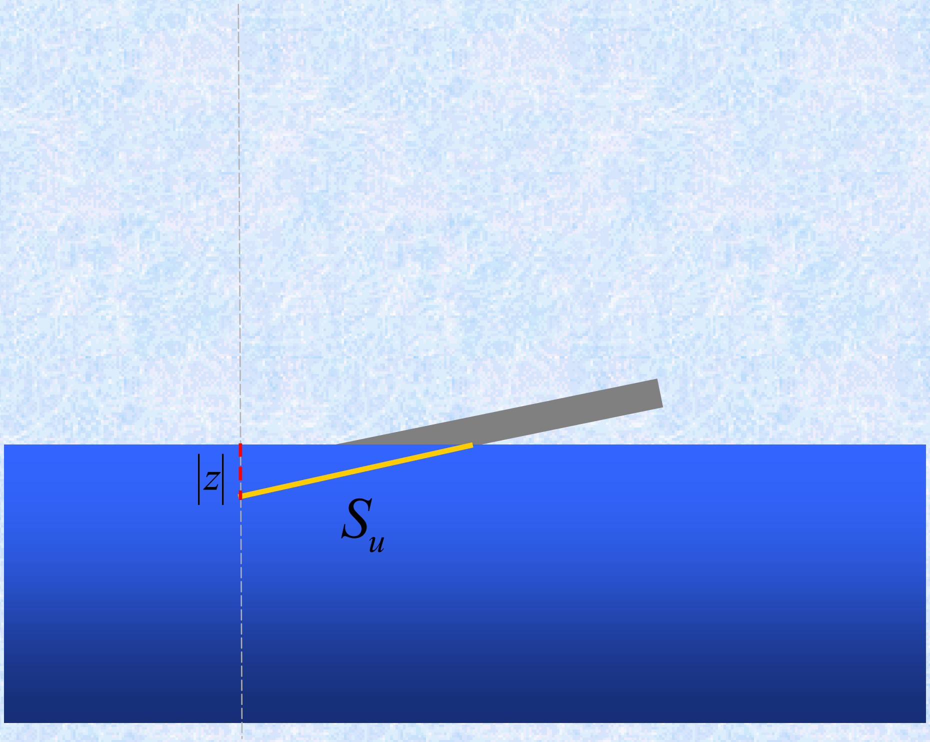
$$\Rightarrow m\frac{d^2 z}{dt^2} = -mg + \frac{1}{2}\rho_w v_{x0}^2 S_{im}(z)\Gamma$$

$$\Gamma \equiv C_d \cos\vartheta - C_l \sin\vartheta$$

- The function *S*(*z*) depends on the shape of the pebble
- The model will use a circular pebble

Circular pebble





$$S_u(z) \approx \frac{|z| r}{\sin \vartheta}$$

r – radius of the pebble

$|z|$ - immerging depth

Estimating the minimal velocity - forces

- Bouncing condition:

$$mg < \overline{F_\perp}$$

$$\Rightarrow mg < \frac{1}{2}\rho_w \overline{S}_{im} v^2 \Gamma$$

$\overline{F_\perp}$ - mean value of vertical component of hydrodinamical force

$\overline{S}_{im}$ - mean value of immerged surface

- For the estimation we may approximately take

$$\overline{S}_{im} \approx \frac{1}{2} r^2 \pi$$

r – pebble radius

$$\Rightarrow v \rangle \frac{2}{r}\sqrt{\frac{mg}{\pi\rho_w\Gamma}}$$

- For our model (20˚ angle of attack) this limit was 4 m/s which is in good agreement with the experimentally obtained value of 3.8 m/s
- For smaller attack angles the velocity is greater

Estimating the minimal velocity - friction

- Another bouncing condition can be found using energy:

$$\frac{1}{2} m v_{x0}^2 \geq W_d$$

W_d – work of friction (drag)

$$W_d = -v_{x0} \int_0^{t_{coll}} F_x(t) dt \quad \Rightarrow W_{tr} \approx -mg\mu v_{x0} t_{coll}$$

t_{coll} – time of pebble collision with water surface

μ - ″coefficient of friction″, def. $\mu \equiv \dfrac{C_d \cos\vartheta + C_l \sin\vartheta}{\Gamma}$

$$\Rightarrow v \approx v_{x0} \rangle 2g\mu t_{coll}$$

- Collision time is generally of the order of magnitude 10^{-1} s
- That means that the condition for 20° angle of attack is

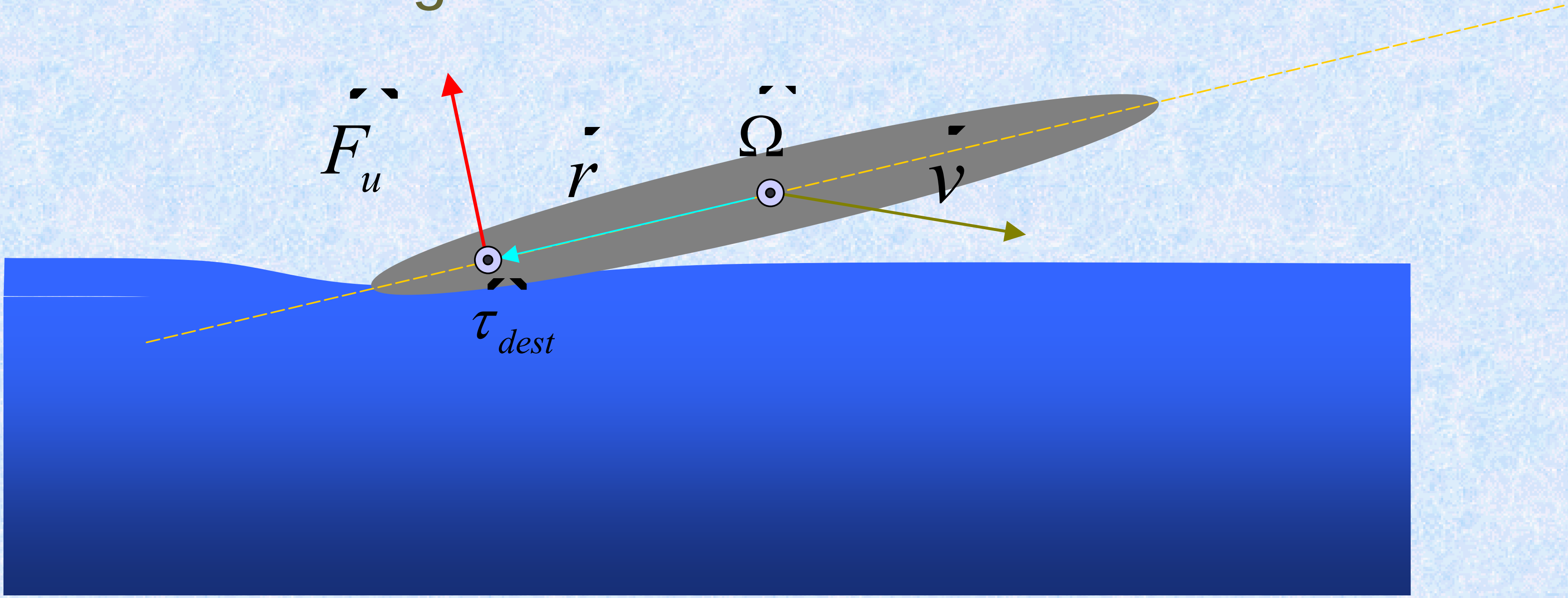
$$v > 3 \text{ m/s}$$

- This condition is less restrictive than the previous, so we can say that the unique condition is

$$\Rightarrow v \rangle \frac{2}{r}\sqrt{\frac{mg}{\pi\rho_w\Gamma}}$$

Why rotating the pebble?

- During the contact of pebble and water surface a destabilizing force occurs:

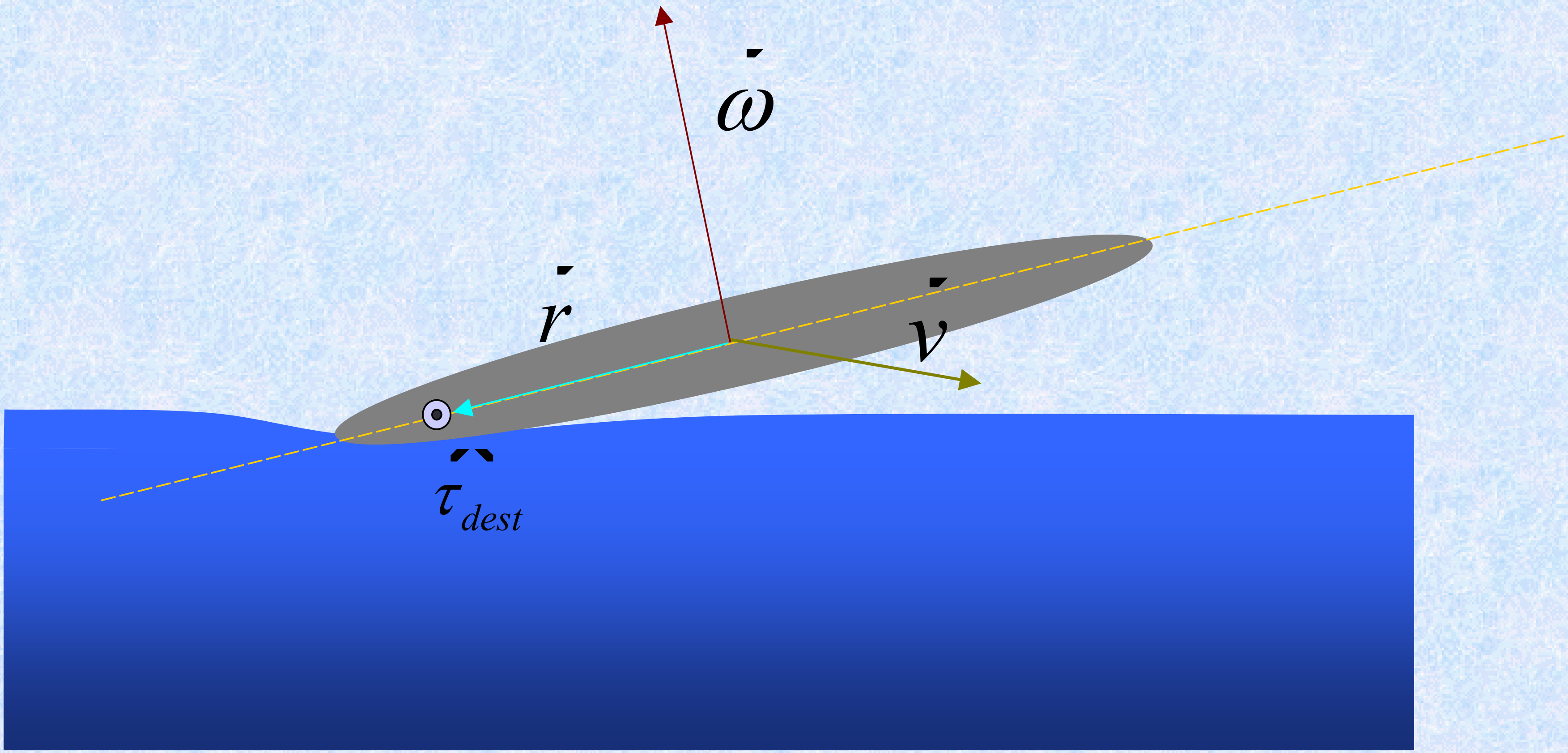


r – radius vector

Ω – angular velocity of precession (changes θ)

τ_{dest} - destabilizing torque

- If the pebble is rotated, the resulting gyroscopic effect will counteract the change of attack angle:



ω – rotational angular velocity

Conclusion

- The conditions needed for a pebble to skip on a water surface are:
  - Initial velocity usually greater than 3 m/s
  - Angle of attack between 10° and 30° (for our model the optimal angle was 20°)
  - Large rotational velocity