11^{th} IYPT '98
solution to the problem no. 3
presented by the team of Finland

Jeremias Seppä

Spinning disc

Investigate and explain the phenomenon of a spinning annular disc as they progress down a straight, cylindrical rod. If the rod is moved upwards at a defined velocity, the disc spins at constant height. Investigate the mechanism.

Abstract

We have studied the spinning motion of an annular ring down a vertical rod. In our studies we have used a video camera and analysis of the torques of the forces acting on the ring. When taking videopictures of the motion we have slowed down the spinning by tilting the rod. The combined action of gravity, friction and normal forces is that the ring, when falling down, simultaneously rotates in a tilted position so that its normal is not in the same plane as the rod. The essential thing is that the ring touches the rod at two points which are on different sides of the plane which goes through the center of the ring.

Overview

- Principal points of the motion of the disc
- Explanation

1 Principal points of the motion of the disc

- The motion of the disc is a combination of a fall and a rotation.
- The torque of the frictional forces between the disc and the rod accelerates and maintains the rotation.
- The torque of the normal forces between the disc and the bar cause the precession of the disc, i.e. the rotation of the direction of the momentary angular momentum vector.
- The general equation $\frac{d\bar{L}}{dt} = \bar{\tau}$ limits the speed of rotation of $\bar{L}$.
- Now the torque $\bar{\tau}$ due to the normal forces increases as the speed of precession increases. This enables continuous increasement of the precession speed and the falling speed of the centre of mass of the disc.
- The increasement of the spinning speed is limited by the resistance forces, i.e. air resistance and the friction on the contact surface between the disc and the rod. Both of them increase as the speed of rotation increases.
- Thus, the motion of the disc is rolling down around the rod whereby the centre of mass of the disc, as well as the contact points between the disc and the rod move down along spiral lines.

2 Explanation

The dots stand for the contact points of the disc and the rod. (figure 1)

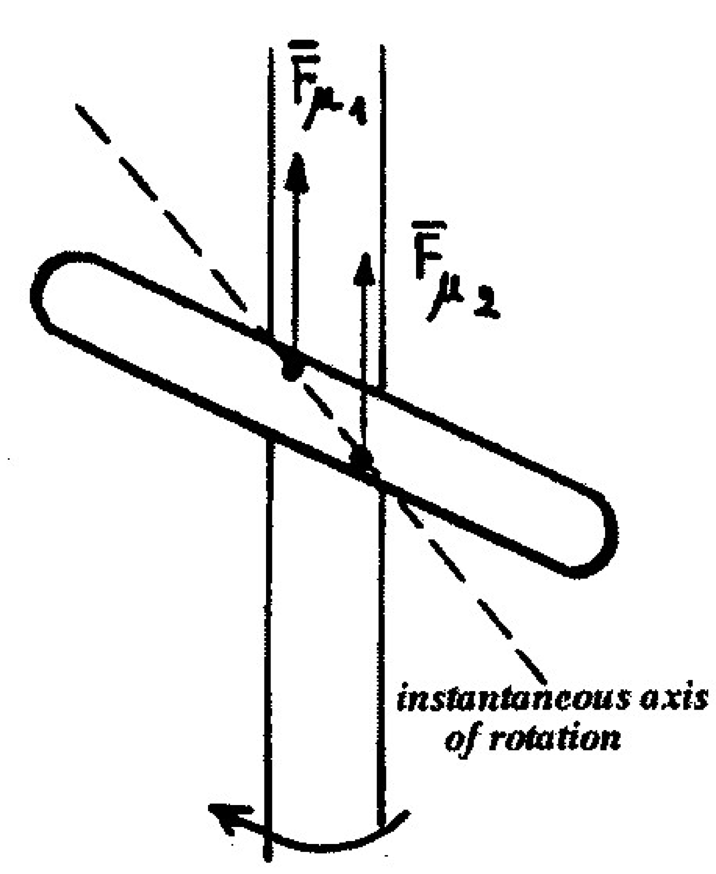

Figure 1: Contact points

The torques of frictional and normal forces with respect to the centre of mass of the disc. (figure 2)

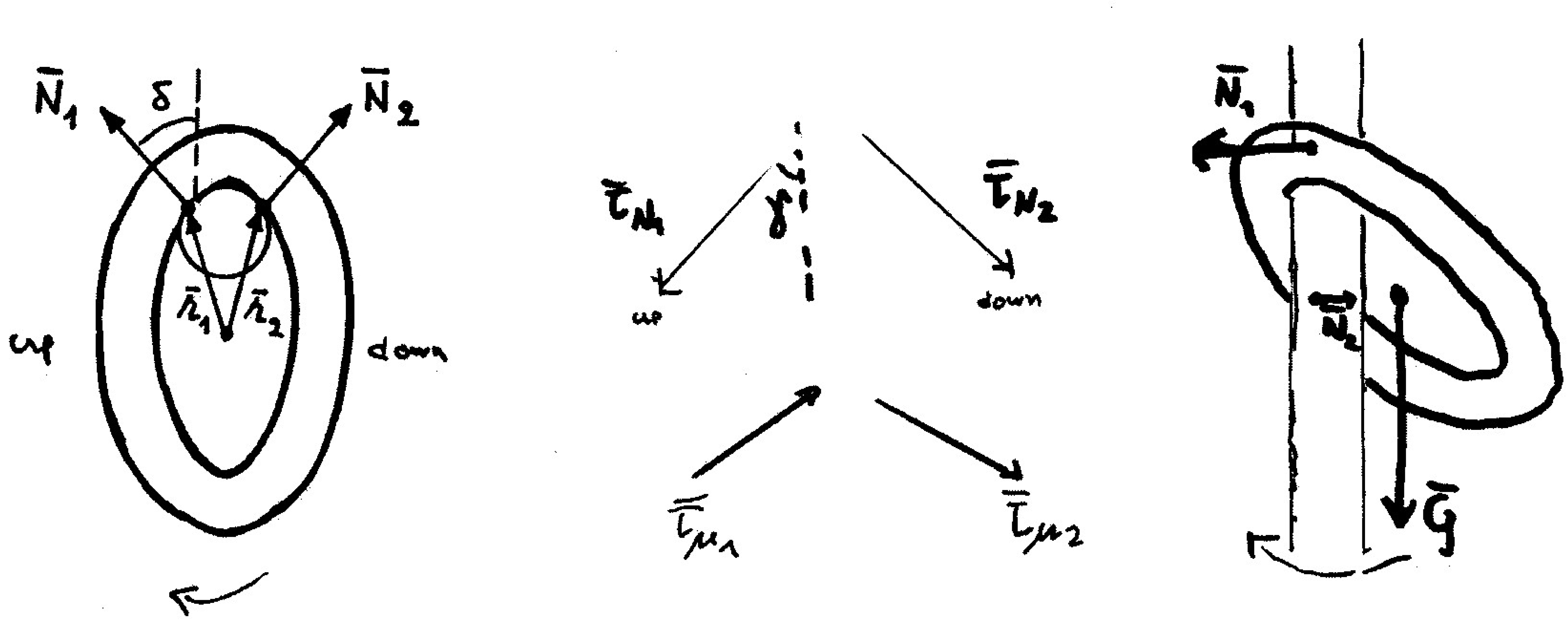

Figure 2: Forces

The net torques of the frictional and normal forces. (figure 3)
The torque of friction is decomposed into two components which are parallel and perpendicular to the actual axis of rotation. (figure 4)

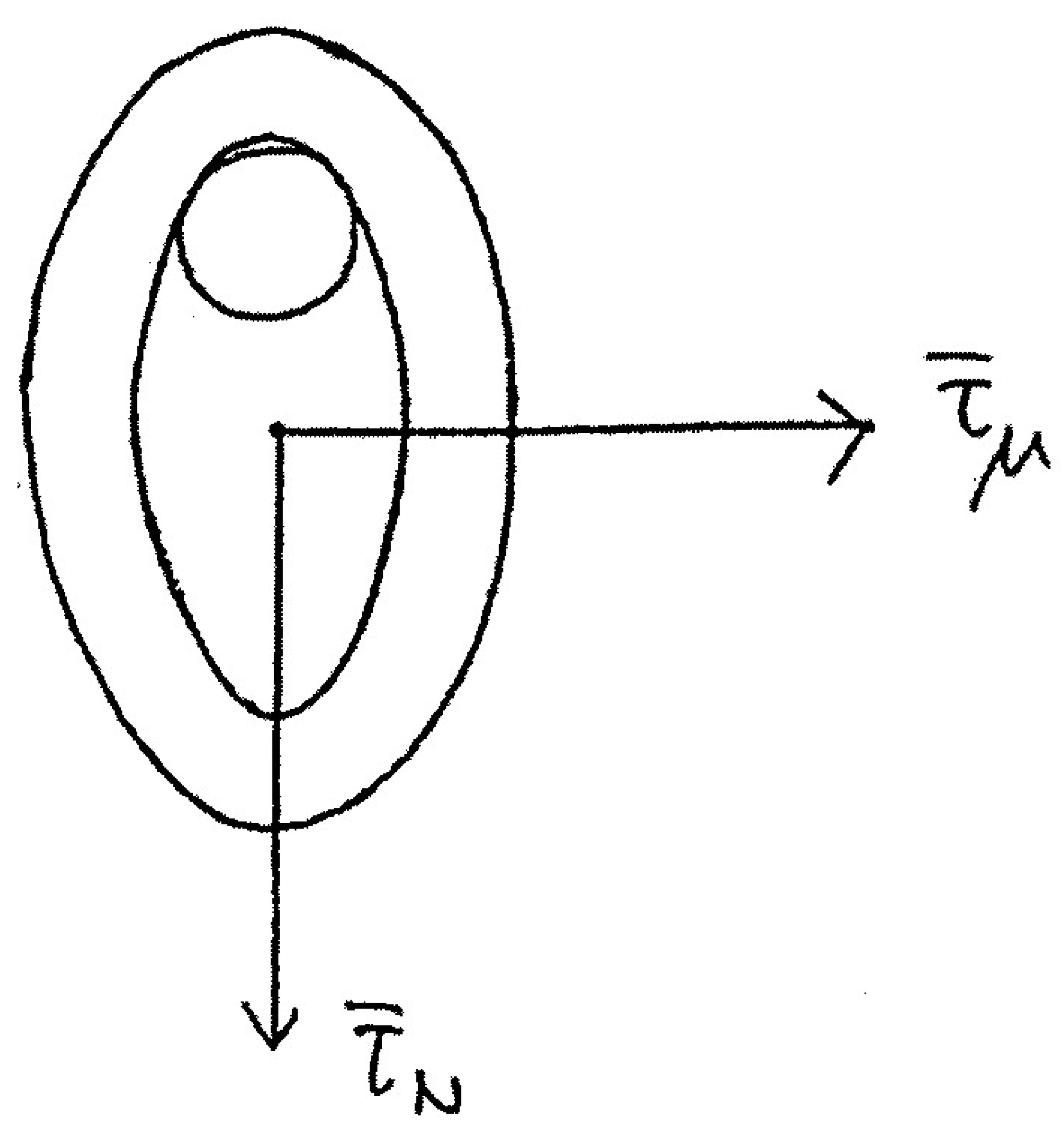

Figure 3: Net torques

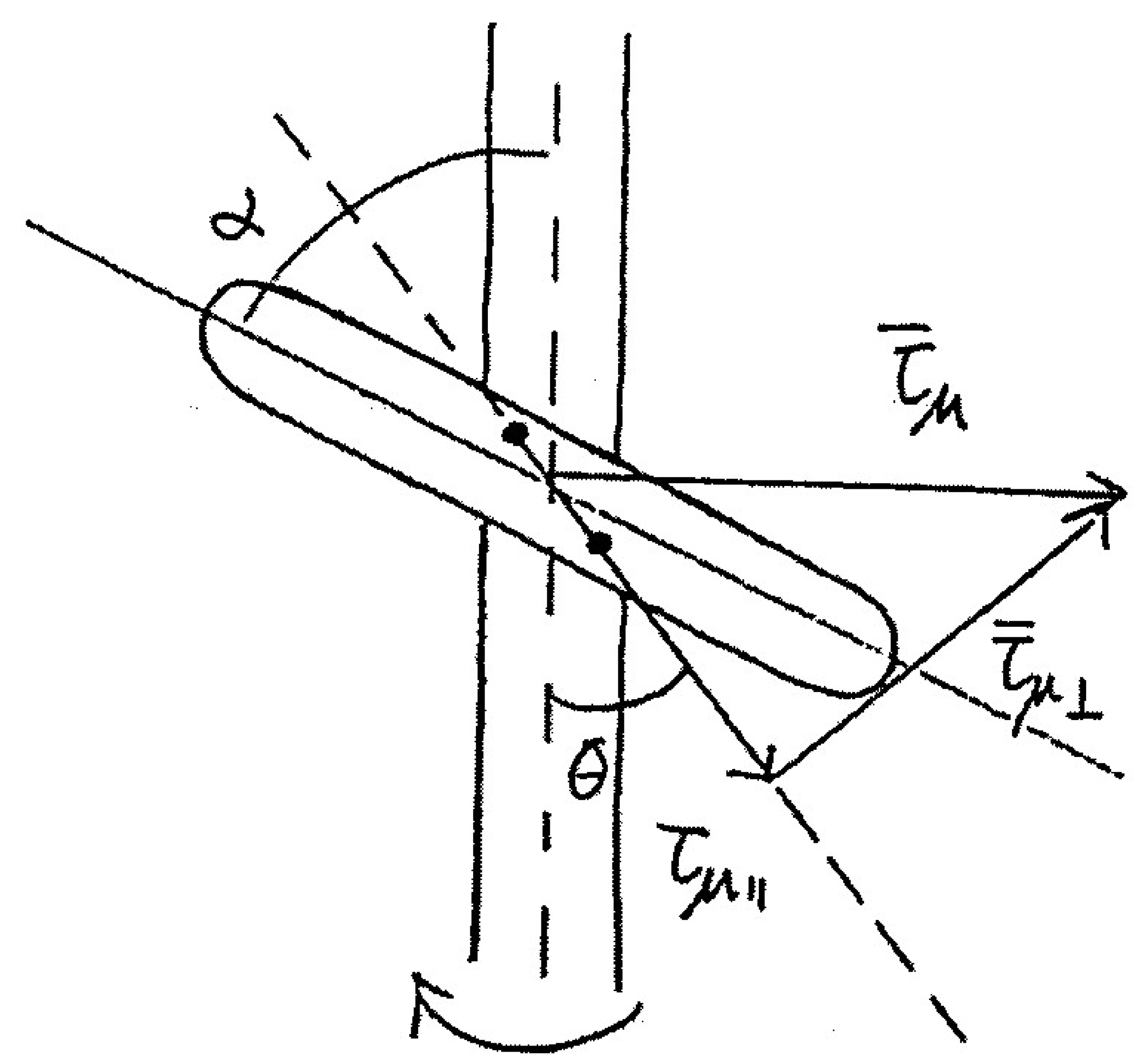

Figure 4: Net torques

$\bar{\tau}_{\mu\parallel}$ accelerates the speed rotation.
The friction at the contact points of the disc and the rod cancels the effect of $\bar{\tau}_{\mu\perp}$.
$\bar{\tau}_N = \bar{\tau}_{N_1} + \bar{\tau}_{N_2}$ changes the direction of rotation axis, i.e. is the cause for precession.
Let us assume that there is no gliding between the disc and the rod. So the two contact points move down along a helical path. (figure 5)

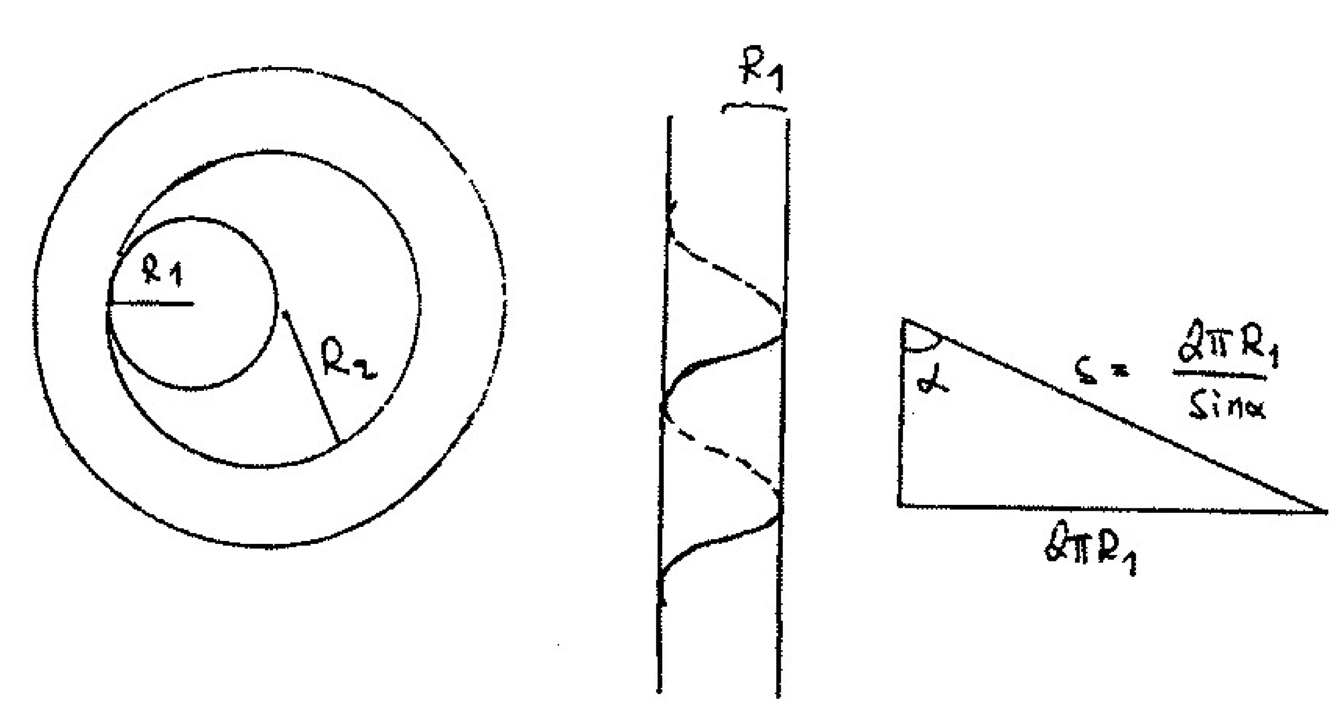

Figure 5: Gliding

α is the angle between the axis the rod and the plane of the disc.
During one revolution around the rod the disc rotates the distance $s = \frac{2\pi R_1}{\sin\alpha}$.

ω = the angular velocity around the rod
ω_p = the angular velocity around the centre of gravity

$$\omega_p = \frac{R_1}{R_2 \sin\alpha}\omega$$

The condition for a rolling motion: $F_\mu^s > G$ or $\mu_s N > G$,
where μ_s is the static coefficient of friction

$$\begin{aligned} N &= ma_n \\ &= \frac{mv^2}{r} \\ \Rightarrow \mu_s \frac{mv^2}{r} &> mg \\ v^2 &> \frac{rg}{\mu_s} \end{aligned}$$

By substitutions $v = r\omega$ and $r = (R_2 - R_1)\sin\alpha$:

$$\omega > \sqrt{\frac{g}{\mu_s(R_1 - R_2)\sin\alpha}}$$

The equation of motion for rotation of disc:

$$\frac{d\bar{L}}{dt} = \bar{\tau} \text{ and } \bar{\tau} = \sum \bar{r} \times \bar{F}$$

As the instantaneous axis of rotation of the disc is not a symmetry axis of the disc, $\bar{L}$ is not in the direction the axis of rotation (figure 6).

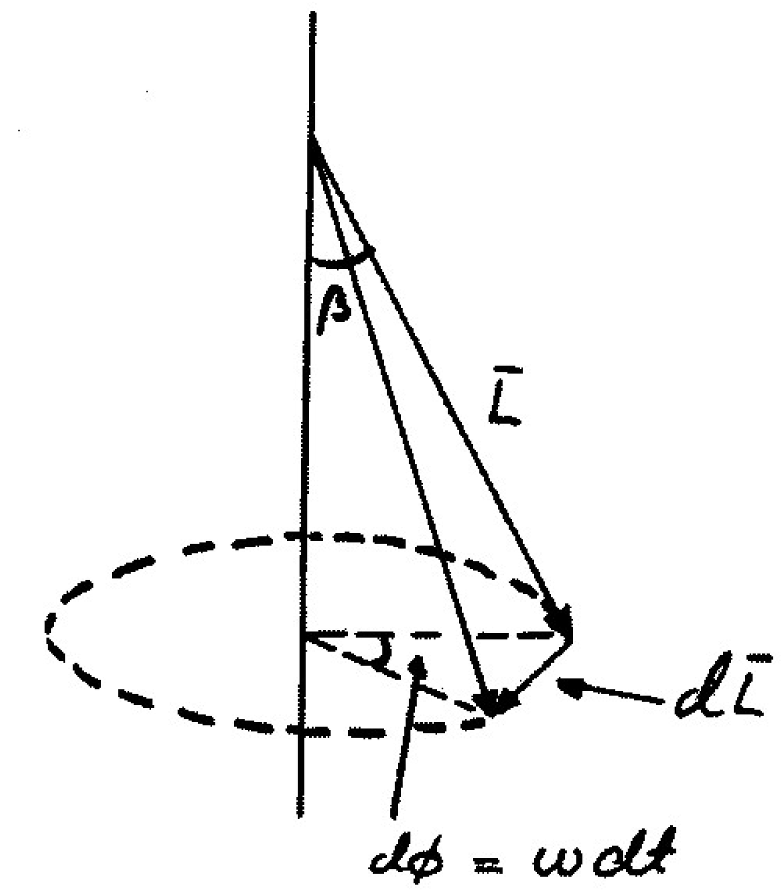

Figure 6: Rotation

$$\begin{aligned}
dL &= L\sin\beta d\Phi \\
&= L\sin\beta\omega dt \\
\tau &= \tau_N \\
\tau_N &= |\bar{\tau}_{N_1} + \bar{\tau}_{N_2}| \\
\frac{dL}{dt} &= \tau_N \\
L\sin\beta\omega &= \tau_N \\
L &= I\omega_R \\
N &= |\bar{N}_1 + \bar{N}_2| \\
\tau_N &= |\bar{\tau}_{N_1} + \bar{\tau}_{N_2}| \\
&= (\tau_{N_1} + \tau_{N_2})\cos\gamma \\
&= 2\tau_{N_1}\cos\gamma \\
\bar{\tau}N_1 &= \bar{R}_2 \times \bar{N}_1 \\
&= R_2N_1\sin\left(\bar{R}_2;\bar{N}_1\right) \\
\Rightarrow I\omega\omega_R\sin\beta &= 2R_2N_1\sin\left(\bar{R}_2;\bar{N}_1\right)\cos\gamma \\
N &= N_{1_\perp} + N_{2_\perp} \qquad \text{(perpendicular component to the axis)} \\
N &= \frac{mv^2}{R_2 - R - 1} \\
&= m(R_2 - R_1)\sin\alpha\omega^2 \\
N_{1_\perp} &= N_1\cos\delta \\
N_1 &= \frac{N_{1_\perp}}{\cos\delta} \\
&= \frac{N}{2\cos\delta} \\
&= \frac{m(R_2 - R_1)\sin\alpha\omega^2}{2\cos\delta} \text{ by substituting } \uparrow \\
I\omega\omega_R\sin\beta &= 2R_2\frac{m(R_2 - R_1)\sin\alpha\omega^2}{2\cos\delta}\sin\left(R_2;\bar{N}_1\right)\cos\gamma \\
\omega_R &= \frac{R_2m(R_2 - R_1)\sin\alpha\sin\left(\bar{R}_2;\bar{N}_1\right)\cos\gamma}{I\sin\beta\cos\delta}\omega
\end{aligned}$$