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solution to the problem no. 5
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Water jet

If a vertical water jet falls down onto a horizontal plate, standing waves will develop on the surface of the jet. Investigate the dependence of this phenomenon on different parameters.

Overview

- Theory
  - Flow velocity
  - Capillary waves
- Extension

1 Theory

1.1 Flow velocity

When the jet is leaving the orifice of the tap, it has the cross section A. Because of gravity effects, the water will be accelerated. This leads to a higher velocity of the flow and therefore a smaller cross section. Consequently the surface energy will decrease.
Neglecting surface tension, the relation can be formulated as follows:

$$\frac{1}{2}mv^2 = \frac{1}{2}mv_0^2 + mgh$$

The equation of continuity says:

$$\pi r^2 v = \pi r_0^2 v_0$$

This yields:

$$r(h) = \frac{r_0}{\sqrt[4]{1 + \frac{2gh}{v_0^2}}}$$

This equation can be verified experimentally for example by projecting the water jet and measuring the picture (see figure 1). In our experiments, the values measured were in good correspondence with the one predicted in the theory. Still the theory can be accurated by considering surface tension.

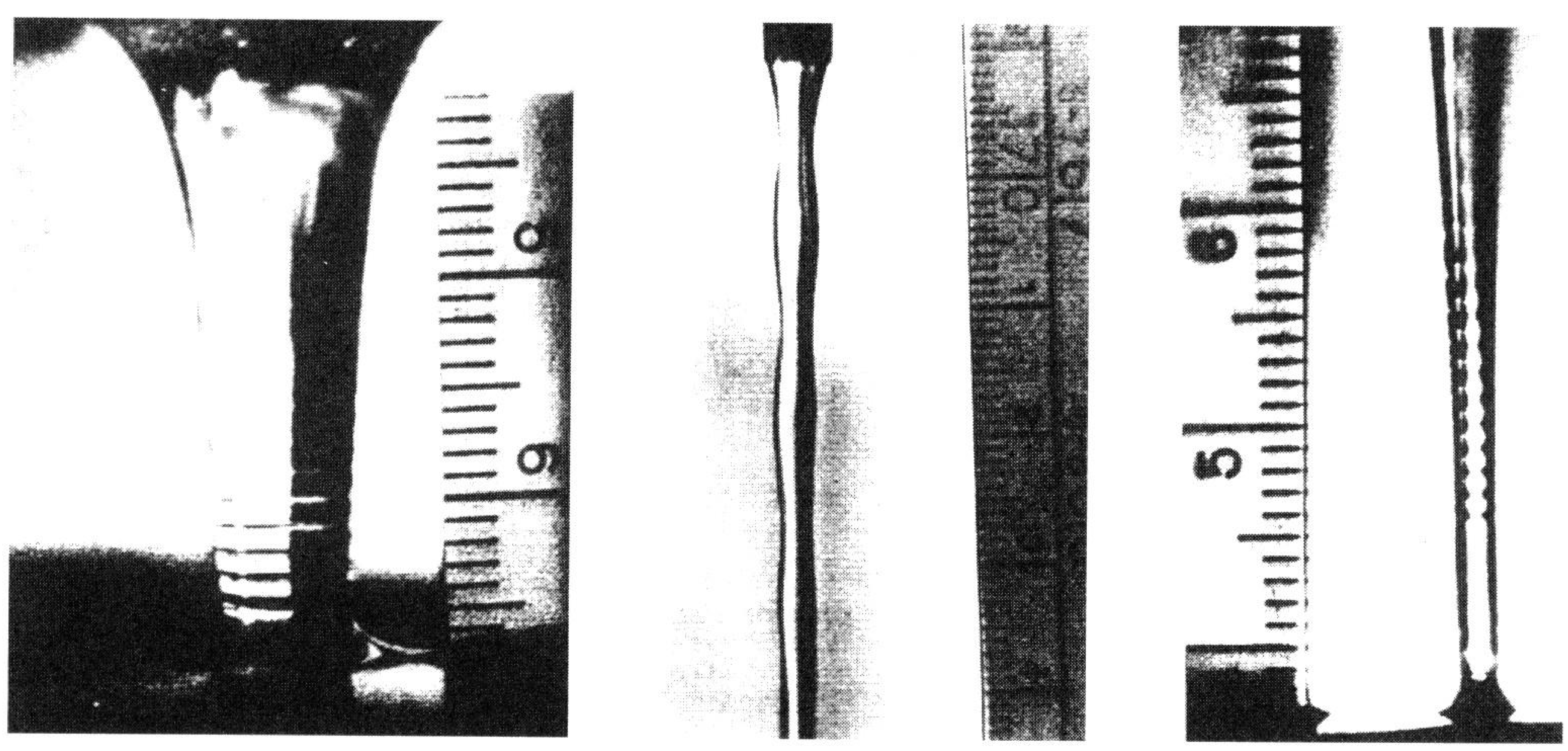

Figure 1: Pictures of the experiment

Considering surface tension you get this equation, which cannot be solved algebraically:

$$\frac{1}{2}mv^2 + 2\pi r\sigma v dt = \frac{1}{2}mv_0^2 + mgh + 2\pi r_0 \sigma v_0 dt \qquad \left(\sigma = 72.75 \cdot 10^{-3}\frac{\mathrm{N}}{\mathrm{m}}\right)$$

Using this equation and the equation of continuity you can calculate the velocity of the jet at a certain height h.

1.2 Capillary waves

The waves which occur on the surface of the water jet, are capillary waves. This means that they mostly depend on the surface tension. The amplitude is small compared to the radius of the jet. If the amplitude was larger, the water jet would become unstable and finally split into drops.
The phase velocity for capillary waves on a plane surface can be calculated as follows:

$$v_{\mathrm{ph}} = \sqrt{\frac{2\pi\sigma}{\rho\lambda}}$$

This equation is valid for plane capillary waves but also, in good approximation, for the present type of waves. Using the wave number $k = \frac{2\pi}{\lambda}$ this can be written as:

$$v_{\text{ph}} = \sqrt{\frac{\sigma k}{\rho}}$$

At the source of the wave, waves with different wavelengths are generated. The wavelengths lie in a small band around

$$\lambda = \frac{2\pi\sigma}{\rho v^2}$$

where v is the velocity of the jet at the height h below the orifice.
The group velocity of this band can be calculated by differentiating v_{ph}.

$$v_{\text{gr}} = \frac{d\omega}{dk} = \frac{3}{2} v_{\text{ph}} = \frac{3}{2}\sqrt{\frac{\sigma k}{\rho}}$$

The wavelength also depends on the height.
The points of the jet, when the maxima can be observed, are those, where the waves interference constructively. So, at these points, the phase velocity of one wavelength in the small band must equal the jet velocity.
From the equation of continuity, the dependence of the jet velocity on the height can be calculated:

$$\begin{aligned} \pi r^2 v &= \pi r_0^2 v_0 \\ \frac{r_0^2}{\sqrt{1 + \frac{2gh}{v_0^2}}} v &= r_0^2 v_0 \\ v(h) &= \sqrt{v_0^2 + 2gh} \end{aligned}$$

If this expression is used in the equation for the wavelength, it yields:

$$\lambda = \frac{s\pi\sigma}{\rho\left(v_0^2 + 2gh\right)}$$

From this equation it can also can be noticed, that the wavelength doesn't actually depend directly on the radius of the jet.

2 Extension

The above equation for the dispersion relation is valid for plane capillary waves. Extending this to waves on a cylindrical surface, it can be shown that:

$$\omega^2 = \frac{\gamma k}{\rho R^2} \frac{I_1(kR)}{I_0(kR)} \left(k^2 R^2 - 1\right)$$

where I_0 and I_1 are modified Bessel functions. As these functions are more complicated, the dispersion relations for the plane waves has been used. It fits the experimental data very well (about 10% discrepancy).

Using Laplace equation:

$$\Phi_{rr} + \frac{1}{r}\Phi_r + \Phi_{zz} = 0$$

in the range $0 \leq r \leq a + \eta$ this yields:

$$\Phi_r = \eta_t + \Phi_z \eta_z$$

and using Bernoulli's equation:

$$\frac{p}{\rho} + \Phi_t + \frac{1}{2}\left(\Phi_z^2 + \Phi_r^2\right) = 0$$

You obtain

$$\frac{\sigma}{\rho a} - \Phi_t - \frac{1}{2}\left(\Phi_z^2 + \Phi_r^2\right) = \frac{\sigma}{\rho\sqrt{1+\eta_z^2}}\left(\frac{1}{r} - \frac{\eta_{zz}}{1+\eta_z^2}\right)$$

considering that there is pressure discontinuity at the surface given by:

$$p - p_a = \sigma\left(\frac{1}{R_1} + \frac{1}{R_2}\right)$$

where p_a is the external pressure, p the pressure just beneath the jet surface, R_1 and R_2 the principal radii of curvature of the jet surface.

$$\begin{aligned} \Phi_t &= \eta_t \\ \Phi_t &= \frac{\sigma}{\rho}\left(\frac{\eta}{a^2} + \eta_{zz}\right) \end{aligned}$$

For $r = a$ instead of $r = a + \eta$:

$$\begin{aligned} \eta(z,t) &= \eta_0 \cos\left(kz - \Omega t\right) \\ \Phi(r,z,t) &= \frac{\Omega\eta_0}{kI_1(ka)} I_0(kr) \sin\left(kz - \Omega t\right) \end{aligned}$$

with dispersion relation:

$$\omega^2 = \frac{1}{\beta}\frac{I_1(\alpha)}{I_0(\alpha)}\alpha\left(\alpha^2 - 1\right)$$