III. Solution of the problems for the 18th IYPT

1. PROBLEM № 2: THE TWO BALLS PROBLEM

SOLUTION OF BRAZIL

Problem № 2: The Two Balls Problem
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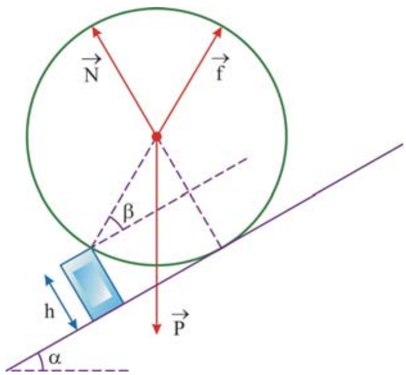
The problem

Two balls placed in contact on a tilted groove sometimes do not roll down. Explain the phenomenon and find the conditions, under which it occurs.

Physic Insight:

First, we will treat about the case where we have just one sphere in a plan. If we want the sphere does not roll down, we need to fix an obstacle (as showed in the picture) in front of it.



Picture 1: A ball in a ramp with an obstacle

In this case we have $\vec{f}$ force: contact force that obstacle exert in the sphere (with *R* radius), $\vec{P}$: sphere's weight, $\vec{N}$:Normal force that the titled ramp exert in the sphere; *h* is the obstacle height fixed in the ramp, α is the ramp's inclination and β is the angle formed between the ramp and $\vec{f}$ direction.
In the equilibrium, the external force sum at the sphere must be equal to zero (null). Thus,

$$\sum_i F_{ix} = \sum_i F_{iy} = 0 \qquad (1)$$

where F_{ix} and F_{iy} are the *x* and *y* component of the i-esimal force, respectively. In this way:

- X axis: $f \cdot \cos(\alpha + \beta) = N \cdot \sin\alpha$
- Y axis: $P = f \cdot \sin(\alpha + \beta) + N \cdot \cos\alpha$

Solving these two equations (in function of *P*) we have:

$$f = \frac{\sin\alpha}{\cos\beta} \cdot P \quad (2) \qquad\qquad N = \frac{\cos(\alpha + \beta)}{\cos\beta} \cdot P \quad (3)$$

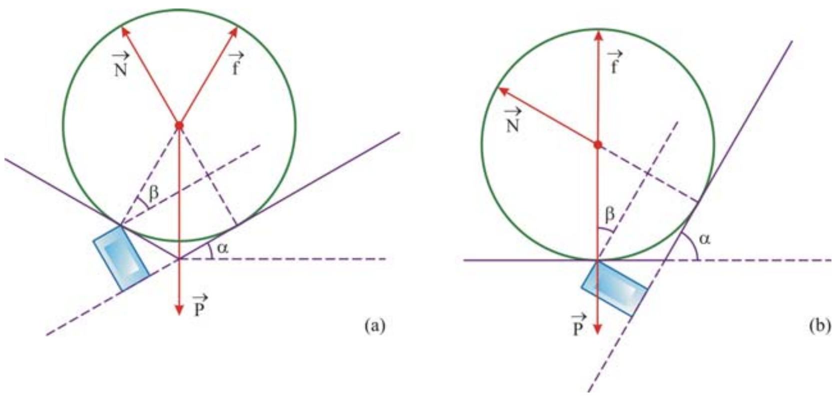
This solution above just have a physical sense, when $N \geq 0$. Thus, from equation 3 we obtain:

$$\alpha + \beta \leq \frac{\pi}{2} \quad (4)$$

With this, we can conclude that the contact point between the sphere and the obstacle must stay at the left of $\vec{P}$ vector; otherwise the sphere will roll over the obstacle.

We could thing in other way to understand the equilibrium: let's imagine that the ball is placed in a groove with V shape (picture 2a). Therefore, we can consider that the ball is in under a tilted ramp with an obstacle as the spotted lines in the drawing shows.

The equilibrium configuration happens when both walls of the groove are inclined upwards (stability). It is way, in picture 2b, the ball is under a permanent equilibrium configuration (instability), which means that the sphere can roll to left in one moment. (It is possible because the left groove wall is in a horizontal position). Geometrically, this condition happens when $\alpha + \beta = \pi/2$. In this way we also arrive in the equilibrium condition (equation 4).



Picture 2: Equilibrium configurations

With assistance of this theory, we can substitute the obstacle for a small sphere, placed in the same local: at the left of the superior ball. The two balls would not roll down because the friction force torque over them must be compensates by the torque of the contact force between them. Therefore, we can expect that the torques cancel themselves and do not occur any rolling movement.

We can notice that if the left sphere is bigger than the right sphere, it will be impossible maintaining the equilibrium. It would be equivalent to having the obstacle at the right side of the ball. So, we would not deal with this situation.

The Two Balls Problem:

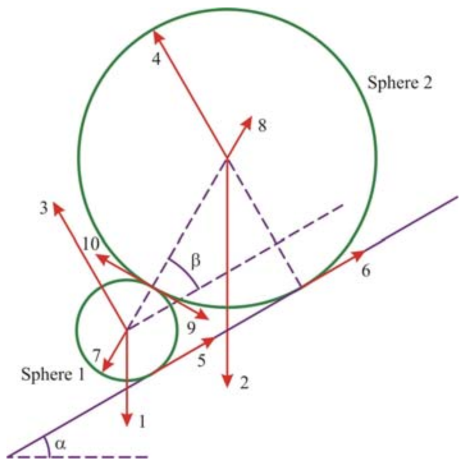
We need to analyze the condition in under which two balls in touch in a tilted groove do not roll down.

Picture 3 shows us the two balls in a tilted ramp and the external forces that are applied over each ball.

The ball at the left is *sphere* 1 and the ball at the right is *sphere* 2.

These forces are:

1. Sphere 1 weight force: $\vec{P}_1$;
2. Sphere 2 weight force: $\vec{P}_2$;
3. Sphere 1 Normal: $\vec{N}_1$;
4. Sphere 2 Normal: $\vec{N}_2$;
5. Friction force in sphere 1: $\vec{f}_1$;
6. Friction force in sphere 2: $\vec{f}_2$;
7. Sphere 2 contact force in sphere 1, which tends to push down the sphere e 1: $\vec{f}_3$
8. Reaction to force $\vec{f}_3$: - $\vec{f}_3$;
9. Sphere 2 contact force in sphere 1, which tends to rotate sphere 1 in a clockwise movement: $\vec{f}_4$;
10. Reaction to force $\vec{f}_4$: - $\vec{f}_4$



Picture 3: The two balls problem

the angle α represents the ramp inclination and β represents the angle formed between the tilted ramp and the forces pair $\vec{f}_3$ and $\vec{f}_4$ actuation line

As we want that the system continues in equilibrium, the sum of the forces over each spheres have to be zero.

$$\sum_i F_{ix} = \sum_i F_{iy} = 0$$

So, such as the external forces, as well the external torques which acts in the two balls must annul themselves. First, the force equilibrium equations for each ball are:

Sphere 1:

- X axis: $f_1 \cos\alpha + f_4 \sin(\alpha+\beta) = f_3 \cos(\alpha+\beta) + N_1 \sin\alpha$;
- Y axis: $\vec{P}_1 + f_3 \sin(\alpha+\beta) + f_4 \cos(\alpha+\beta) = f_1 \sin\alpha + N_1 \cos\alpha$;

- Rotation: $f_1 = f_4$

<u>Sphere 2:</u>

- X axis: $f_2 \cos\alpha + f_3 \cos(\alpha+\beta) = f_4 \sin(\alpha+\beta) + N_2 \sin\alpha$;
- Y axis: $\vec{P}_2 = f_2 \sin\alpha + f_3 \sin(\alpha+\beta) + f_4 \cos(\alpha+\beta) + N_2 \cos\alpha$;
- Rotation: $f_2 = f_4$

The system solution is showed below:

$$N_1 = P_1 \cos\alpha + \frac{\sin\alpha}{2\cos\beta}\left[P_1 + P_2 + (P_2 - P_1)\sin\beta\right],$$

$$N_2 = P_2 \cos\alpha - \frac{\sin\alpha}{2\cos\beta}\left[P_1 + P_2 + (P_2 - P_1)\sin\beta\right],$$

$$f_3 = \frac{\sin\alpha}{2\cos\beta}\left[P_2 - P_1 + (P_2 + P_1)\sin\beta\right],$$

$$f_1 = f_2 = f_4 = \frac{1}{2}\sin\alpha(P_1 + P_2).$$

In order not roll any ball, it is necessary that

$$f_1 \le \mu_1 N_1 \quad (6), \qquad f_2 \le \mu_2 N_2 \quad (7), \qquad f_4 \le \mu_3 f_3 \quad (8)$$

where μ_1, μ_2 μ_3 are the static friction coefficient between the sphere 1 and the ramp, between the sphere 2 an the ramp and between sphere 1 and sphere 2, respectively. The system will be in movement imminence as soon as one of these equations becomes saturated.

Now, we must study the module of $\alpha, \beta, P_1, P_2, \mu_1, \mu_2, \mu_3$ that saturate the equilibrium's conditions (equation 6, 7, 8) with purpose of finding out the imminence conditions in which the spheres roll down.

Firstly, let's analyze the restriction 8. So,

$$\cos\beta - \mu_3 \sin\beta \le \mu_3 \frac{P_2 + P_1}{P_2 - P_1} \quad (9)$$

which solution is:

$$\cos\beta \ge \frac{\mu_3 p + \sqrt{\mu_3^{\,2} + (1 - p^2)\mu_3^{\,4}}}{1 - \mu_3^{\,2}},$$

where $p = \dfrac{P_2 - P_1}{P_2 + P_1}$. We must to emphasize that is not the unique solution. There are negative solutions which refers to case in where the smallest ball is at the right side.

As we know that:

$$\beta_{\min} \le \beta \le \frac{\pi}{2} - \alpha \quad (10)$$

we have:

$$\beta_{\min} = \arccos \frac{\mu_3 p + \sqrt{\mu_3^{\ 2} + (1 - p^2)\mu_3^{\ 4}}}{1 - \mu_3^{\ 2}}$$

In an analog form, to the equation 7:

$$\cot\alpha \ge \frac{P_1 + P_2}{2P_2}\left(\frac{1}{\mu_2} + \frac{1 + p\sin\beta}{\cos\beta}\right) \qquad (11)$$

In the same form, to equation 6:

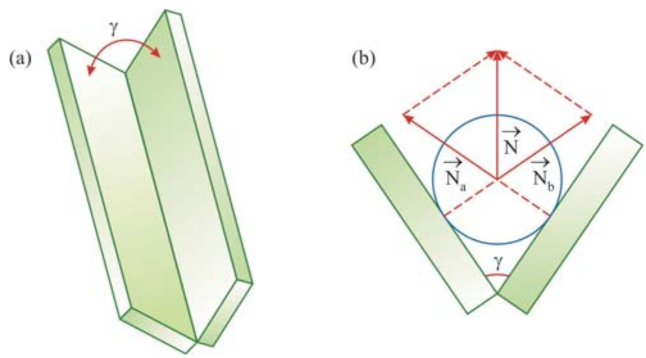
$$\cot\alpha \ge \frac{P_1 + P_2}{2P_2}\left(\frac{1}{\mu_1} - \frac{1 + p\sin\beta}{\cos\beta}\right) \qquad (12)$$

The restriction ensemble (10), (11) e (12) defines α and β modules, in function of the others parameters $P_1, P_2, \mu_1, \mu_2, \mu_3$

<u>Groove:</u>

The first point we want to discuss is the tilted ramp (opening groove wall's angle = 180º). Two balls in a tilted ramp always roll down because the bi-dimensional configuration is unstable. Let's remember that we are working with the hypothesis that the spheres are confined to move along the line which defines the tilted plan (conform picture 3).

The experiment in a tilted plan would just be successful if we worked with cylinders, instead of balls. As we want to respect the problem, using spheres, we must have to use a gutter with a V format (the groove), as it is showed in picture 4.



Picture 4: Groove's characteristic

The difference between this model and the model utilized in the previous section is the β angle's determination. We can notice that the equilibrium force equations remain unaltered. For example, the Normal force vector which acts over a sphere in this groove (see picture 4b) remains in the rolling plan. The transversal component to the vectors $\vec{N}_a$ and $\vec{N}_b$ rolling plan cancel themselves (see picture 4b) and yet remains in the rolling plan. β angle's determination in the assembly of picture 3 is extremely simple: $\sin\beta = (R-r)/(R+r)$. In the groove, the β angle's determination should depend on γ. As illustrated in picture 4b, the distance between the groove basis (intersection point between the two walls) and the sphere centre (with radius equal to *R*) is:

$$D = \frac{R}{\sin\frac{\gamma}{2}}$$

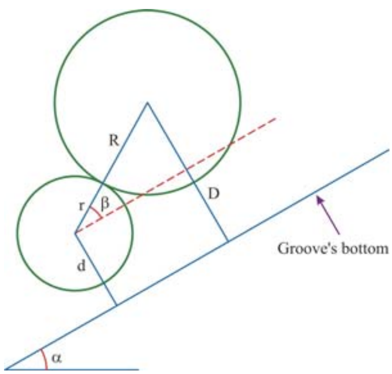
In this way, β angle have to be determined by (see picture 5),

$$\sin\beta = \frac{R-r}{(R+r)\sin\frac{\gamma}{2}} \quad \textbf{(16)}$$

There is a minimum γ. It is necessary to remember condition $\alpha+\beta \le \frac{\pi}{2}$, therefore:

$$\frac{R-r}{(R+r)\sin\frac{\gamma}{2}} \le \cos\alpha, \Rightarrow \sin\frac{\gamma}{2} \ge \frac{R-r}{(R+r)\cos\alpha}$$

To finish, the experiment to be done still have seven free parameters: $\alpha, \beta, P_1, P_2, \mu_1, \mu_2, \mu_3$. In the true, we can simplify it to just six parameters because in the restriction inequations (6), (7), (8) P_1 and P_2 appears in the two sides, so, the interest quantity is the relative mass between the spheres, and not the absolute mass of each one.



Picture 5: A lateral view of the groove

In the attached Excel plan *two_balls* we manipulated all the formulas, where we can place the variables (theoretical values or experimental data) to obtain the maximum angle for a specific situation. The angle formed between the two groove walls is fixed (in our case 90º). We fixed this angle in order to a better experimental performance. However, as we fixed this angle, we can not apply it into a future theoretical angle's maximum model.

As we are working with an opening groove's angle equal to 90º, equation 13 becomes:

$$\sin\beta = \frac{R-r}{(R+r)\sin\frac{\pi}{4}}$$

We are going to use its plan to compare the experiment with theory and see the difference between these two parts of the problem's resolution

<u>Not rolling conditions</u>:
In this section we will determine the conditions and the inclination angle α which must satisfy the equations presented, in order to occur the phenomenon. It means that α must satisfy:

$$\alpha_{\min} \le \alpha \le \alpha_{\max}$$

where $\alpha_{\min}$ and $\alpha_{\max}$ depends on $\beta, P_1, P_2, \mu_1, \mu_2, \mu_3$.

$\alpha_{\min}$ determination is simple: in the case the groove is in the horizontal position, the system will stay in rest, hence, $\alpha_{\min} = 0$ for any of the others values.

$\alpha_{\max}$ determination is more complex: it demands that all restrictions (4), (6), (7) and (8) is satisfied. From condition 4:

$$\alpha_1 = \frac{\pi}{2} - \beta \qquad (13)$$

where α_1 is a possible $\alpha_{\max}$ value, and, by inequation 10, $\beta \ge \beta_{\min}$.

If $\beta < \beta_{\min}$, $\alpha_{\max} = 0$.

It is important to notice that condition if the sphere at the left is smaller than the sphere at the right; otherwise β would be smaller or equal to zero.

The others possible $\alpha_{\max}$ values come from inequation (11) and (12) and they are, respectively:

$$\alpha_2 = \arctan\frac{2P_2\mu_2\cos\beta}{(P_1+P_2)(\cos\beta+\mu_2+\mu_2 p\sin\beta)} \qquad (14)$$

$$\alpha_3 = \arctan\frac{2P_1\mu_1\cos\beta}{(P_1+P_2)(\cos\beta-\mu_1-\mu_1 p\sin\beta)} \qquad (15)$$

Therefore,

$$\alpha_{\max} = \min\{\alpha_1, \alpha_2, \alpha_3\}$$

For determining the maximum inclination that the groove can has, without the spheres rolling down, we must calculate angles $\alpha_1, \alpha_2, \alpha_3, \beta_{\min}$. If $\beta \geq \beta_{\min}$, then the maximum groove inclination will be given by the smaller value among $\alpha_1, \alpha_2, \alpha_3$. Otherwise, If $\beta < \beta_{\min}$, the phenomenon (problem) will not occur.

This conclusion ends up the study about phenomenon, but they do not give a good indication how to vary $\beta, P_1, P_2, \mu_1, \mu_2, \mu_3$ parameters in the way that it maximizes the groove inclination. It happens because the equation which determines $\alpha_1, \alpha_2, \alpha_3$ are completely non-linear equation.

First, we need to understand more about the opening wall's angle (which is always, in our case, 90º)

In the next topics we will explain the manipulation possibilities of these equations in one more simplified context.

<u>Simplified Situation:</u>

In this section we will treat about the realization possibilities of the described phenomenon in situations more simplified. Our simplification choice will be specified in each case

- Same Size Spheres

Considering two spheres with same size (same radius), is it possible that they do not roll down?

In this situation, β is equal to zero (see equation 13). In this way:

$$f_3 = \frac{\sin\alpha}{2}(P_2 - P_1) \qquad\qquad f_4 = \frac{\sin\alpha}{2}(P_1 + P_2)$$

As $f_4 \leq \mu_3 f_3$,
$$\mu_3 \geq \frac{P_2 + P_1}{P_2 - P_1} > 1$$

Hardly we will find in the nature material which have static friction coefficient bigger than 1. So we conclude that the experiment success in this situation is remote (if it is not impossible).

We confirm our hypothesis supported in the excel plan *two_balls*.

- Sphere and groove made of the same material

Now we will analyze the case in which we have both spheres and groove made of the same material (e. g. wood). This imply in $\mu_1 = \mu_2 = \mu_3 = \mu$. Defining Δ as the relation between the spheres weight ($P_2 = \Delta P_1$), the minimum β is given by:

$$\beta_{\min} = \arccos\frac{\mu(1-\Delta) + \mu\sqrt{(1+\Delta)^2 + 4\Delta\mu^2}}{(1+\Delta)(1+\mu^2)}$$

We will test this equation experimentally and see the difference between the theoretical value and the experimental one.

Experiment:
Our experiment consists in reproducing the problem according to the explanation giving. After it, we intend to place the experimental data in the theory equation (previous studied) and see how much approximate is our experience.

We made a groove and fixed an angle's measure equipment at its base. Photos of the apparatus are attached. Our variables, in this experiments, were:

<u>The surface material</u>: cardboard paper, polystyrene paper, paper, wood or plastic.

<u>The balls were made of</u>: polystyrene, plastic, leather, steel, glass.

In our experiment, we choose two balls and one surface. We measure the maximum inclination angle (α). After this, we compared with the maximum alfa's angle given by equations 13, 14 or 15 (we can use this equation because we know the relation between the two ball radius, and, consequently, β). The theoretical values were taken from the Excel plan.

We repeat the experience changing the ball combinations and marking the respective angles.

As we said before, in our experiment the angle between the walls of the groove have not been changed (as we used two fixed wood board as our groove, the angle was 90º angle)

The first step in the experiment was measuring the characteristic of the ball (for substituting in the equations), like the diameter, and consequently, the radius, the mass, and, consequently, the weight

For measuring the mass, we made use of a balance (with maximum precision as a hundredth of the gram) and for measuring the diameter we used special ruler (with maximum precision as a tenth of the millimeter).

The balance's error is 0,01g (it is written in the technical characteristic protocol)

The measure instrument error is the half of the smallest measure step. So, the error in the diameter was 0,005mm

The experimental measure data as well the correspondent error is attached in the <u>Excel plan</u> *two_balls_measures*.

It is important to emphasis that we did not work just with full solid balls: we worked also with spherical husks (ball with air inside).

Completely solid (6):	Golf, iron 1 and 2, glass 1 and 2, plastic ball
Spherical husk (12):	All polystyrene balls, tennis balls, blue, orange and violet balls, ping-pong balls

The unique difference between this two kind of ball (hollow or full) is the great difference between the weight force in balls with almost the same size.

As we need to place the static friction coefficient to calculate theoretically the maximum angle, we need to know its value for each material combination in the experiment.

A complete static friction coefficient table is attached in a Word document *two_balls_friction*

Those coefficients we did not managed to find in any table, we calculate using the equation below (where θ is the movement imminence angle).

$$\mu_{static} = \tan\theta$$

Just in four material's combination we did not have a theory value we made this experience. The results are showed in the table below

	θ	tan θ = μ (static)
Cardboard and polystyrene	27º	0,50
Plastic and polystyrene	24,5º	0,45
Paper and polystyrene	20,5º	0,37
Leather and plastic	32º	0,62

DEDUCTIONS

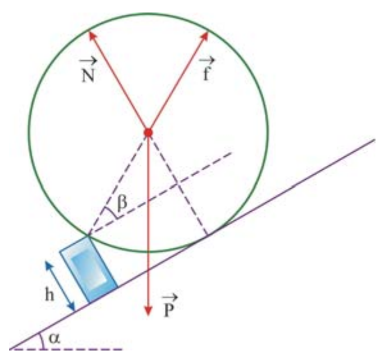
DEDUCTION 1

- X axis: $f \cdot \cos(\alpha + \beta) = N \cdot \sin\alpha$ (I)
- Y axis: $W = f \cdot \sin(\alpha + \beta) + N \cdot \cos\alpha$ (II)

As, $\sum_i F_{ix} = \sum_i F_{iy} = 0$

(I) $N = \dfrac{f \cdot \cos(\alpha + \beta)}{\sin\alpha}$

(II) $N = \dfrac{W - f \cdot \sin(\alpha + \beta)}{\cos\alpha}$



Picture 1: A ball in a ramp with an obstacle

$N = N \rightarrow \dfrac{f \cdot \cos(\alpha+\beta)}{\sin\alpha} = \dfrac{W - f \cdot \sin(\alpha+\beta)}{\cos\alpha}$

□ $f \cdot \cos(\alpha+\beta) \cdot \cos\alpha = \left[W - f \cdot \sin(\alpha+\beta)\right] \cdot \sin\alpha$

□ $f \cdot \cos(\alpha+\beta) \cdot \cos\alpha + f \cdot \sin(\alpha+\beta) \cdot \sin\alpha = W \cdot \sin\alpha$

□ $f \cdot \left[\cos(\alpha+\beta) \cdot \cos\alpha + \sin(\alpha+\beta) \cdot \sin\alpha\right] = W \cdot \sin\alpha$

□ $f \cdot \left[\cos(\alpha+\beta-\alpha\right] = W \cdot \sin\alpha$

□ $f \cdot \left[\cos(\not\alpha+\beta-\not\alpha\right] = W \cdot \sin\alpha$

□ $f \cdot (\cos\beta) = W \cdot \sin\alpha \rightarrow \boxed{f = \dfrac{\sin\alpha}{\cos\beta} \cdot W}$

Placing ***f*** result in equation (I), we have:

$N \cdot \sin\alpha = f \cdot \cos(\alpha+\beta) \rightarrow N \cdot \sin\alpha = \dfrac{\sin\alpha}{\cos\beta} W \cdot \cos(\alpha+\beta) \rightarrow \boxed{N = \dfrac{\cos(\alpha+\beta)}{\cos\beta} \cdot W}$

Deduction of *Alfa*$_1$

$\boxed{N = \dfrac{\cos(\alpha+\beta)}{\cos\beta} \cdot W}$ $\quad N \geq 0 \quad$ $\alpha+\beta \leq \dfrac{\pi}{2}$ $\quad \boxed{\alpha_1 = \dfrac{\pi}{2} - \beta}$

Deduction of *Alfa*$_2$

For finding β minimum, we need to pick condition

$$f_1 \leq \mu_1 N_1$$

We need to substitute ***f***$_1$, ***N***$_1$ and ***μ***$_1$ (static friction coefficient between the two ball's material). Therefore,

$$f_1 = \frac{1}{2}\sin\alpha(W_1 + W_2)$$

$$N_1 = W_1\cos\alpha + \frac{\sin\alpha}{2\cos\beta}\left[W_1 + W_2 + (W_2 - W_1)\sin\beta\right]$$

□ $f_1 \leq \mu_1 N_1$

□ $\dfrac{1}{2}\sin\alpha(W_1 + W_2) \leq \mu_1\left[W_1\cos\alpha + \dfrac{\sin\alpha}{2\cos\beta}\left[W_1 + W_2 + (W_2 - W_1)\sin\beta\right]\right]$

□ $$\frac{\frac{1}{2}\sin\alpha(W_1+W_2)\le\mu_1\left[W_1\cos\alpha+\frac{\sin\alpha}{2\cos\beta}\left[W_1+W_2+\left(W_2-W_1\right)\sin\beta\right]\right]}{\frac{\sin\alpha}{2}}$$

□ $$(W_1+W_2)\le\mu_1\left[\frac{2W_1\cos\alpha}{\sin\alpha}+\frac{\left[(W_1+W_2)+\left(W_2-W_1\right)\sin\beta\right]}{\cos\beta}\right]$$ and

$$\frac{\cos\alpha}{\sin\alpha}=\cot\alpha \rightarrow$$

□ $$(W_1+W_2)\le\mu_1\cdot 2W_1\cot\alpha+\frac{\mu_1(W_1+W_2)}{\cos\beta}+\frac{\mu_1\left(W_2-W_1\right)\sin\beta}{\cos\beta}$$

and $w=\dfrac{W_2-W_1}{W_2+W_1}$ $\rightarrow w\cdot(W_2+W_1)=W_2-W_1$

□ $$(W_1+W_2)-\frac{\mu_1(W_1+W_2)}{\cos\beta}-\frac{\mu_1\left(w\cdot(W_2+W_1)\right)\sin\beta}{\cos\beta}\le\mu_1\cdot 2W_1\cot\alpha \rightarrow$$

□ $$\frac{(W_1+W_2)\cos\beta}{\mu_1\cos\beta}-\frac{\mu_1(W_1+W_2)}{\mu_1\cos\beta}-\frac{\mu_1 w\cdot(W_2+W_1)\sin\beta}{\mu_1\cos\beta}\le 2W_1\cot\alpha \rightarrow$$

□ $$(W_1+W_2)\left(\frac{\cos\beta}{\mu_1\cos\beta}-\frac{\mu_1}{\mu_1\cos\beta}-\frac{\mu_1 w\cdot\sin\beta}{\mu_1\cos\beta}\right)\le 2W_1\cot\alpha$$

□ $$\frac{(W_1+W_2)}{2W_1}\left(\frac{1}{\mu_1}-\frac{1}{\cos\beta}-\frac{w\cdot\sin\beta}{\cos\beta}\right)\le\cot\alpha \rightarrow$$

$$\frac{(W_1+W_2)}{2W_1}\left(\frac{1}{\mu_1}-\frac{1+W\cdot\sin\beta}{\cos\beta}\right)\le\cot\alpha$$

□ $$\boxed{\cot\alpha\ge\frac{W_1+W_2}{2W_1}\left(\frac{1}{\mu_1}-\frac{1+w\sin\beta}{\cos\beta}\right)}$$

$$\cot\alpha=\frac{1}{\tan\alpha} \rightarrow \frac{1}{\tan\alpha}\ge\frac{W_1+W_2}{2W_1}\left(\frac{1}{\mu_1}-\frac{1+w\sin\beta}{\cos\beta}\right)$$

□ $$\frac{1}{\frac{W_1+W_2}{2W_1}\left(\frac{1}{\mu_1}-\frac{1+w\sin\beta}{\cos\beta}\right)} \geq \tan\alpha$$ →

$$\frac{2W_1}{(W_1+W_2)\left(\frac{\cos\beta-\mu_1-\mu_1 w\sin\beta}{\mu_1\cos\beta}\right)} \geq \tan\alpha$$

□ $$\tan\alpha \leq \frac{2W_1\mu_1\cos\beta}{(W_1+W_2)(\cos\beta-\mu_1-\mu_1 w\sin\beta)}$$

□ $$\boxed{\alpha_3 = \arctan\frac{2W_1\mu_1\cos\beta}{(W_1+W_2)(\cos\beta-\mu_1-\mu_1 w\sin\beta)}}$$

Deduction of *Alfa*₃

For finding β minimum, we need to pick condition

$$f_2 \leq \mu_2 N_2$$

We need to substitute $\boldsymbol{f_2}$, $\boldsymbol{N_2}$ and $\boldsymbol{\mu_2}$ (static friction coefficient between the two ball's material). Therefore,

$$f_2 = \frac{1}{2}\sin\alpha(W_1+W_2) \qquad N_2 = W_2\cos\alpha - \frac{\sin\alpha}{2\cos\beta}\left[W_1+W_2+\left(W_2-W_1\right)\sin\beta\right]$$

□ $$f_2 \leq \mu_2 N_2$$

□ $$\frac{1}{2}\sin\alpha(W_1+W_2) \leq \mu_2\left[W_2\cos\alpha - \frac{\sin\alpha}{2\cos\beta}\left[W_1+W_2+\left(W_2-W_1\right)\sin\beta\right]\right]$$

□ $$\frac{\frac{1}{2}\sin\alpha(W_1+W_2) \leq \mu_2\left[W_2\cos\alpha - \frac{\sin\alpha}{2\cos\beta}\left[W_1+W_2+\left(W_2-W_1\right)\sin\beta\right]\right]}{\frac{\sin\alpha}{2}}$$

□ $$(W_1+W_2) \leq \mu_2\left[\frac{2W_2\cos\alpha}{\sin\alpha} - \frac{\left[(W_1+W_2)+\left(W_2-W_1\right)\sin\beta\right]}{\cos\beta}\right]$$ and $$\frac{\cos\alpha}{\sin\alpha} = \cot\alpha$$

→

□ $$(W_1+W_2) \leq \mu_2\cdot 2W_2\cot\alpha - \frac{\mu_2(W_1+W_2)}{\cos\beta} - \frac{\mu_2\left(W_2-W_1\right)\sin\beta}{\cos\beta}$$

and $w = \dfrac{W_2 - W_1}{W_2 + W_1}$ → $w \cdot (W_2 + W_1) = W_2 - W_1$

□ $(W_1 + W_2) + \dfrac{\mu_2 (W_1 + W_2)}{\cos\beta} + \dfrac{\mu_2 \left(w \cdot (W_2 + W_1)\right) \sin\beta}{\cos\beta} \le \mu_2 \cdot 2W_2 \cot\alpha$ →

□ $\dfrac{(W_1 + W_2)\cos\beta}{\mu_2 \cos\beta} + \dfrac{\mu_2 (W_1 + W_2)}{\mu_2 \cos\beta} + \dfrac{\mu_2 w \cdot (W_2 + W_1)\sin\beta}{\mu_2 \cos\beta} \le 2W_2 \cot\alpha$ →

□ $(P_1 + P_2)\left(\dfrac{\cos\beta}{\mu_2 \cos\beta} + \dfrac{\mu_2}{\mu_2 \cos\beta} + \dfrac{\mu_2 p \cdot \sin\beta}{\mu_2 \cos\beta}\right) \le 2P_2 \cot\alpha$

□ $\dfrac{(W_1 + W_2)}{2W_2}\left(\dfrac{1}{\mu_2} + \dfrac{1}{\cos\beta} + \dfrac{w \cdot \sin\beta}{\cos\beta}\right) \le \cot\alpha$ →

$\dfrac{(W_1 + W_2)}{2W_2}\left(\dfrac{1}{\mu_2} + \dfrac{1 + w \cdot \sin\beta}{\cos\beta}\right) \le \cot\alpha$

□ $\boxed{\cot\alpha \ge \dfrac{W_1 + W_2}{2W_2}\left(\dfrac{1}{\mu_2} + \dfrac{1 + w\sin\beta}{\cos\beta}\right)}$

$\cot\alpha = \dfrac{1}{\tan\alpha}$ → $\dfrac{1}{\tan\alpha} \ge \dfrac{W_1 + W_2}{2W_2}\left(\dfrac{1}{\mu_2} + \dfrac{1 + w\sin\beta}{\cos\beta}\right)$

□ $\dfrac{1}{\dfrac{W_1 + W_2}{2W_2}\left(\dfrac{1}{\mu_2} + \dfrac{1 + w\sin\beta}{\cos\beta}\right)} \ge \tan\alpha$ →

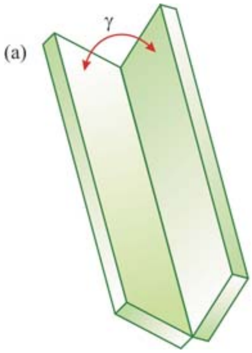
$\dfrac{2W_2}{(W_1 + W_2)\left(\dfrac{\cos\beta + W_2 + \mu_2 w \sin\beta}{\mu_2 \cos\beta}\right)} \ge \tan\alpha$

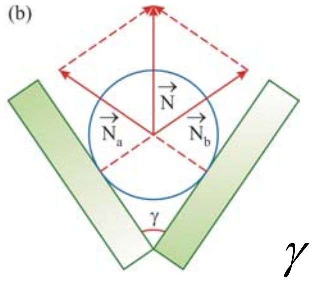
□ $\tan\alpha \le \dfrac{2W_2 \mu_2 \cos\beta}{(W_1 + W_2)(\cos\beta + \mu_2 + \mu_2 w \sin\beta)}$

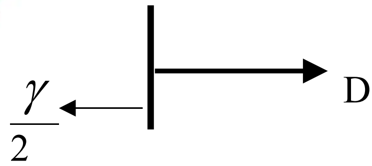
□ $\boxed{\alpha_2 = \arctan \dfrac{2W_2 \mu_2 \cos\beta}{(W_1 + W_2)(\cos\beta + \mu_2 + \mu_2 w \sin\beta)}}$

Deduction of Beta

- In relation to gama (γ) and to the ball's size -

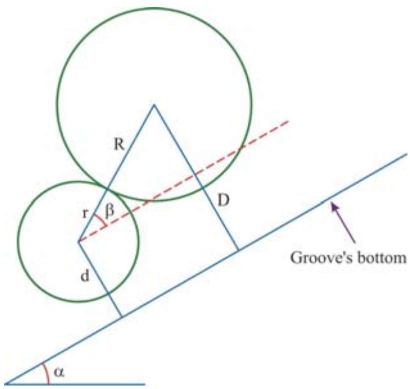






Groove: **frontal view**

From the rectangle triangle, we have: $D = \dfrac{R}{\sin \frac{\gamma}{2}}$ and $d = \dfrac{r}{\sin \frac{\gamma}{2}}$



□ $\sin \beta = \dfrac{D-d}{R+r}$ → $\sin \beta = \dfrac{\dfrac{R}{\sin \frac{\gamma}{2}} - \dfrac{r}{\sin \frac{\gamma}{2}}}{R+r}$ →

$\sin \beta = \dfrac{(R-r)}{R+r}\left(\dfrac{1}{\sin \frac{\gamma}{2}}\right)$ → $\sin \beta = \dfrac{R-r}{(R+r)\sin \frac{\gamma}{2}}$

□ $\beta = \arcsin \dfrac{R-r}{(R+r)\sin \frac{\gamma}{2}}$

Groove: lateral view

Deduction of *Beta*$_{\text{min}}$

For finding β minimum, we need to pick condition

$$f_4 \le \mu_3 f_3$$

We need to substitute $\boldsymbol{f_4}$, $\boldsymbol{f_3}$ and $\boldsymbol{\mu_3}$ (static friction coefficient between the two ball's material). Therefore,

$f_4 = \dfrac{1}{2}\sin \alpha (W_1 + W_2)$ $\quad f_3 = \dfrac{\sin \alpha}{2\cos \beta}\left[W_2 - W_1 + (W_2 + W_1)\sin \beta\right]$ $\quad f_4 \le \mu_3 f_3$

□ $\dfrac{1}{2}\sin \alpha (W_1 + W_2) \le \mu_3 \left[\dfrac{\sin \alpha}{2\cos \beta}\left[W_2 - W_1 + (W_2 + W_1)\sin \beta\right]\right]$

□ $(W_1 + W_2) \leq \mu_3 \left[\frac{1}{\cos\beta} \left[W_2 - W_1 + (W_2 + W_1)\sin\beta \right] \right]$

□ $\cos\beta \leq \mu_3 \left[\frac{1}{(W_1 + W_2)} \left[(W_2 - W_1) + (W_2 + W_1)\sin\beta \right] \right]$

□ $\cos\beta \leq \mu_3 \left[\frac{(W_2 - W_1)}{(W_1 + W_2)} + \frac{(W_2 - W_1)\sin\beta}{(W_1 + W_2)} \right]$ and $w = \frac{W_2 - W_1}{W_2 + W_1}$

□ $\cos\beta \leq \mu_3 [w + \sin\beta]$ → $\cos\beta \leq \mu_3 w + \mu_3 \sin\beta$

□ $\cos\beta - \mu_3 w \leq \mu_3 \sin\beta$ and $\sin\beta = \sqrt{1 - \cos^2\beta}$

□ $\cos\beta - \mu_3 w \leq \mu_3 \sqrt{1 - \cos^2\beta}$ → $(\cos\beta - \mu_3 w)^2 \leq \left(\mu_3 \sqrt{1 - \cos^2\beta}\right)^2$

□ $\cos^2\beta - 2\mu_3 w \cos\beta + \mu_3{}^2 w^2 \leq \mu_3{}^2 (1 - \cos^2\beta)$

□ $\cos^2\beta (1 + \mu_3{}^2) - \cos\beta \cdot 2\mu_3 w + \mu_3{}^2 (w^2 - 1) \leq 0$ (second class equation)

$a = (1 + \mu_3{}^2)$ / $b = -2\mu_3 w$ / $c = \mu_3{}^2 (w^2 - 1)$

□ If $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ → $\cos\beta \leq \frac{-(-2w\mu_3) \pm \sqrt{(-2w\mu_3)^2 - 4(w^2 - 1)(1 + \mu_3{}^2)\mu_3{}^2}}{2a}$

□ $\cos\beta \leq \frac{2w\mu_3 + \sqrt{4w^2\mu_3{}^2 - 4(w^2 - 1)(1 + \mu_3{}^2)\mu_3{}^2}}{2(1 + \mu_3{}^2)}$ →

$\cos\beta \leq \frac{\mu_3 p + \sqrt{p^2\mu_3{}^2 - (p^2 - 1)(\mu_3{}^2 + \mu_3{}^4)}}{1 + \mu_3{}^2}$ →

$\cos\beta \leq \frac{\mu_3 w + \sqrt{\left[w^2\mu_3{}^2 - (w^2 - 1)\mu_3{}^2\right] + (1 - w^2)\mu_3{}^4)}}{1 + \mu_3{}^2}$ →

$\cos\beta \geq \frac{\mu_3 w + \sqrt{\mu_3{}^2 + (1 - w^2)\mu_3{}^4}}{1 - \mu_3{}^2}$

As we know that: $\beta_{\min} \leq \beta \leq \frac{\pi}{2} - \alpha$

we have

$$\boxed{\beta_{\min} = \arccos \frac{\mu_3 w + \sqrt{\mu_3{}^2 + (1 - w^2)\mu_3{}^4}}{1 - \mu_3{}^2}}$$