2. PROBLEM № 4: HYDRAULIC JUMP

SOLUTION OF KOREA

Problem № 4: Hydraulic Jump
by Overlapping of Gravitational Wave with Viscous Fluid

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The problem

When a smooth column of water hits a horizontal plane, it flows out radially. At some radius, its height suddenly rises. Investigate the nature of the phenomenon. What happens, if a liquid more viscous than water is used?

We investigated hydraulic jump of a radially spreading film of water originated by column-like jet that falls onto a horizontal plate. The reason of formation was suggested in terms of Froude number and overlapping of gravitational waves upstream and downstream. With volume-flux and momentum-flux constancy, some equations were made which describe the jump. The role of viscosity of fluid was explained by laminar boundary layer flow. In experimental parts, the water depth before and after the jump, and the radius of the jump were measured with the variation of water column radius, spreading speed, and viscosity. The depth could be measured with steel probe which was attached to micrometer and connected to ammeter, using the fact that when the probe touches the water surface the voltage changes suddenly. In jump radius, it increases when the efflux radius and speed increase and kinematic viscosity decreases and well matched with equations made.

Introduction

In normal kitchen sink, we can see very interesting phenomenon called ‘Hydraulic jump’.(See Fig. 1.) This phenomenon has been issued for almost one century [1][2][3], and this can be easily identified in everyday life. However the reason of formation and characteristics of the jump have not been explained fully, and still many studies are conducted on it.

In this paper, the hydraulic jump was explained in terms of overlapping of gravitational wave and especially roll of viscosity to the jump was investigated.

Theoretical background

In theoretical part, we approached to the jump in terms of gravitational wave first, and made equations with volume-flux and momentum-flux constancy.

The reason of formation ; Overlapping of wave

Waves in water can be divided into mainly two waves; gravitational wave, and surface wave. For our fluid, of which depth is shorter than the half of the wavelength of wave made, the wave has the characteristic of gravitational wave. That is, the speed of wave is determined only by the depth of the water. When h is the depth of water,

$$v_{wave} = \sqrt{gh}$$

Froude number Fr is the significant number for fluid, which is the ratio between wave speed and fluid speed.[4] Especially for gravitational wave,

Fig. 1. Sample hydraulic jump in kitchen sink.

$$Fr = \frac{v_{water}}{v_{wave}} = \frac{v}{\sqrt{gh}}$$

From now on, v is the speed of water. The fluid can be divided into 2 regions in terms of the Fr.(See Fig. 2.)

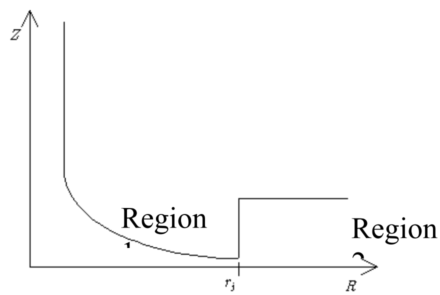

Fig. 2. A sketch of simplified hydraulic jump. Region 1 is supercritical and region 2 is subcritical. r_j means the radius of jump.

Before the jump, in the region 1, Fr is less than 1, and the region 1 is called subcritical. In this region, because the water flows faster than the wave, the wave can go only downstream, but upstream. After the jump, in the region 2, Fr is bigger than 1, and the region 2 is called supercritical. Because the speed of water is slower than that of wave, the wave can go upstream and downstream both now and can be overlapped. At the point of jump, r_j, Fr is equal to 1, which is critical region, and it can be said that the jump position is the point where the wave upstream and downstream can start overlapping of wave. Therefore, the jump was made by the overlapping of gravitational wave.

Hydrodynamic approach

Modeling To make some equations to describe the jump quantitatively, the simplified model of the jump is necessary. In order to make some conditions simple, we made a model like Fig. 3. To make the v_1 constant, we assumed that the depth of water in region 1 decreases until the point of jump. Also, although the jump occurs with some thickness, it was ignored.

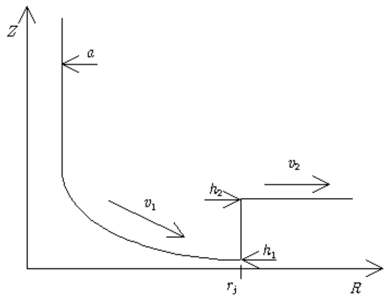

Fig. 3. The modeled hydraulic jump. r_j is the jump radius, a is the radius of the vertical water column. v_1 (v_2) is the speed of the water before the jump(after the jump) and h_1 (h_2) is the depth before the jump(after the jump).

Volume constancy The volume of the water will be preserved because water is incompressible fluid.

$$Q = \pi a^2 v_1 = 2\pi r_j h_1 v_1 = 2\pi r_j h_2 v_2 \quad -(1)\ [5]$$

Then,

$$h_1 v_1 = h_2 v_2 \quad -(2)$$

Momentum constancy Also, the momentum of the stream should be constant. Make a momentum constancy equation by finding the force of the stream in two different ways.

First, consider a cylindrical shell element of fluid from radius r_α to r_β. The total force to deform the element is $F_{tot} = F_\beta - F_\alpha \quad F_\alpha = \frac{d(mv)}{dt} = m\frac{dv}{dt} + v\frac{dm}{dt}$

$\frac{dm}{dt} = \rho Q$ and $\frac{dv}{dt} = 0$ at steady state

$$F_{tot} = (\rho Q v)_\beta - (\rho Q v)_\alpha \quad -(3)$$

Second, find the deforming force due to the pressure.

$$F = \int P dA = 2\pi r \int_0^h \rho g(h-z)dz = \pi r \rho g h^2 \quad -(4)$$

Equating (3) and (4), we have

$(\rho Q v)_\beta - (\rho Q v)_\alpha = \pi r \rho g h_\alpha^2 - \pi r \rho g h_\beta^2$

Using (1) $Q = 2\pi r h v$,

$2\pi r \rho v_\beta^2 h_\beta - 2\pi r \rho v_\alpha^2 h_\alpha = \pi r \rho g h_\alpha^2 - \pi r \rho g h_\beta^2$

$v_\alpha^2 h_\alpha + \frac{1}{2} g h_\alpha^2 = v_\beta^2 h_\beta + \frac{1}{2} g h_\beta^2$,

That is,

$$v_1^2 h_1 + \frac{1}{2} g h_1^2 = v_2^2 h_2 + \frac{1}{2} g h_2^2 \quad \text{– (5)}$$

Depth relationship With two constancy equations, the relationship between h_1 and h_2 can be known easily, and later we will check this relationship is valid with the experimental results.

Use equation (1) to change v_1 and v_2 term in equation (5).

$v_1 = \frac{Q}{2\pi r_j h_1}$, $v_2 = \frac{Q}{2\pi r_j h_2}$

$\frac{Q^2}{4\pi^2 r_j^2 h_1} + \frac{1}{2} g h_1^2 = \frac{Q^2}{4\pi^2 r_j^2 h_2} + \frac{1}{2} g h_2^2 \qquad \frac{Q^2}{4\pi^2 r_j^2}\left(\frac{h_2 - h_1}{h_1 h_2}\right) = \frac{1}{2} g \left(h_2^2 - h_1^2\right)$

Since $h_1 \neq h_2$, and $q \equiv \frac{Q}{2\pi r_j}$ as the volume flux per unit width,

$\frac{q^2}{h_1 h_2} = \frac{1}{2} g \left(h_1 + h_2\right)$

Multiply h_2 to both side and change the equation in terms of h_2.

$\frac{1}{2} g h_1^2 + \frac{1}{2} g h_1 h_2 - \frac{q^2}{h_1} = 0 \qquad h_2 = -\frac{h_1}{2} \pm \frac{h_1}{2}\sqrt{1 + \frac{8q^2}{g h_1^3}}$

Since $h_2 > 0$,

$h_2 = h_1\left(-\frac{1}{2} + \frac{1}{2}\sqrt{1 + \frac{8q^2}{g h_1^3}}\right) = \frac{h_1}{2}\left(-1 + \sqrt{1 + \frac{8q^2}{g h_1^3}}\right)$

$h_2 = \frac{h_1}{2}\left(\sqrt{1 + \frac{8v_1^2}{g h_1}} - 1\right)$ – (6) since $q = \frac{Q}{2\pi r_j} = \frac{2\pi r_j h_1 v_1}{2\pi r_j} = h_1 v_1$

We now found the relationship between h_1 and h_2. However, let's try to simplify more. When $Fr_1 = \frac{v_1}{\sqrt{g h_1}}$, $\frac{h_2}{h_1} = \frac{1}{2}\left(\sqrt{1 + \frac{8v_1^2}{g h_1}} - 1\right) = \frac{1}{2}\left(\sqrt{1 + Fr_1^2} - 1\right)$ – (7)

Now we can check that the our explanation about the hydraulic jump in terms of the Froude number. In the equation (7), when $Fr_1 > 1$, $\frac{h_2}{h_1} > 1$, which means h_2 s bigger than h_1 and when $Fr_1 \leq 1$, $\frac{h_2}{h_1} \leq 1$, which means the jump is not created.

To find out the radius of the jump, use equation (1).

$Q = \pi a^2 v_1 = 2\pi r_j h_1 v_1 = 2\pi r_j h_2 v_2$ – (1)

$h_1 = \dfrac{a^2}{2r_j}$ – (8) $r_j = \dfrac{a^2}{2h_1}$ – (9)

In the equation (5), because $v_1^2 h_1 \gg v_2^2 h_2$ and $\frac{1}{2} g h_2^2 \gg \frac{1}{2} g h_1^2$, then

$$v_1^2 h_1 = \frac{1}{2} g h_2^2$$

and by applying equation (9),

$r_j = \dfrac{v_1^2 a^2}{g h_2^2}$ – (10)

And here, we can make one more $h_1 - h_2$ relationship equation with equation (9) and (10).

$h_2 = \sqrt{\dfrac{2h_1}{g}} v_1$ – (11)

Actually, This is the same results with equation (6) because when $\dfrac{8v_1^2}{gh_1}$ is big enough to ignore the 1,

$$h_2 = \frac{h_1}{2}\left(\sqrt{1 + \frac{8v_1^2}{gh_1}} - 1\right) \approx \frac{h_1}{2}\left(\sqrt{\frac{8v_1^2}{gh_1}} - 1\right)$$

$$\approx \frac{h_1}{2}\left(\sqrt{\frac{8v_1^2}{gh_1}}\right) = \sqrt{\frac{2h_1}{g}} v_1$$

and in our experimental condition, $v_1 = 1.2 m/s$ and $h_1 = 0.4 mm$, $\dfrac{8v_1^2}{gh_1}$ was about 3000, which is big enough.

Roll of viscosity In the problem, the roll of the viscosity of the liquid is asked. The theoretical explanation above does not have any concern of viscosity. It is for the inviscid liquid. For the roll of the viscosity, the boundary layer can be concerned.[6][7][8] The boundary layer means the layer of the water stream which is influenced by the friction with bottom surface and kinematic viscosity of the liquid(See Fig. 4.). Near the bottom, the liquid does not have same speed with the surface; in fact, the speed of the stream is much slower at the bottom. Because there is sudden decrease of the stream at the point of the hydraulic jump, the thickness of the boundary layer and the whole stream becomes same at the point of the jump. The thickness of the viscous laminar boundary layer is

$$\Delta = k\sqrt{\frac{\nu r}{v}} \quad -(12)$$

where k can be experimentally acquired.

For $h \gg \Delta$, the deviation from inviscid flow is negligible. However, as $h \to \Delta$, the no-slip boundary condition becomes important and eventually dominate the whole flow behavior.

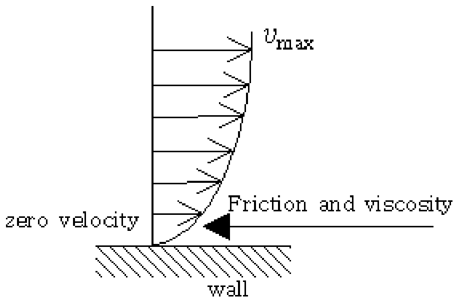

Fig. 4. Side shape of water flow considering laminar boundary layer. At the surface, the speed of water is the biggest.

In our experiment, the Reynolds number is smaller than the transition between laminar and turbulent flow, so we can use the laminar boundary layer.
Consider $h_1 = \Delta$ in the equation (9).

$$r_j = \frac{a^2}{2}\frac{1}{k}\sqrt{\frac{v_1}{\nu r_j}}$$

$$r_j = \frac{a^{4/3}}{(2k)^{2/3}}\left(\frac{v_1}{\nu}\right)^{1/3} = 0.63\frac{a^{4/3}}{k^{2/3}}\left(\frac{v_1}{\nu}\right)^{1/3} \quad -(13)$$

And equation (13) can be very useful because we can know the radius of the jump without the h_1 or h_2 which should be measured to be known. On following experiments, it will be the main equation to compare the experimental data with theory.

Materials & Methods

With the wide water container, the flat board was placed upon the surface of the water and the acryl plate was put on the board.(See Fig. 5.) Between the board and plate, there was plotting paper which made it easier to measure the radius of jump. Next to the water container, there was water reservoir and by the small pump water was sprinkled onto the plate along the plastic tube. The amount of flowing water was controlled by the clamp attached to the end-point of plastic tube. The height of end-point of tube and water reservoir was able to be changed and it changed the efflux speed of water.

Near the place at which the jump was made, the sawn micrometer was set with steel stick tightly fixed by steel stand. At the end point of micrometer, steel pin was attached and it was connected to the ammeter which showed the voltage difference between the pin and the water in water container by putting the other end of ammeter into it. At normal condition the ammeter shows 0V, but by rotating micrometer when the endpoint of pin touches the water surface, the voltage changes drastically because water is not perfect insulator. The surface level could be known by measuring the scale of micrometer when the voltage changes, and the bottom level could be known by rotating the micrometer continuously until the endpoint of pin touches the bottom plate. By this method, it was able to measure the depth of water flow in 10 unit of length.

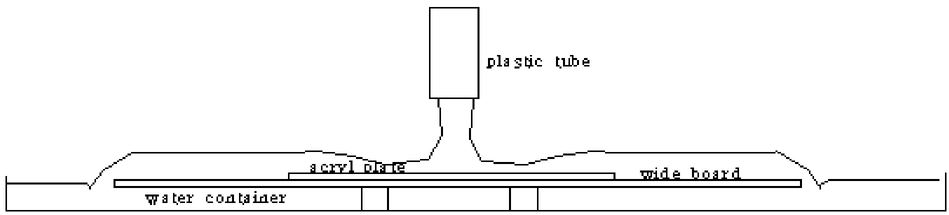

Fig. 5. Side shape of simplified experimental setup. Plastic tube was connected to water reservoir and pump. Between acryl plate and board, the plotting paper was put.

With three variables, efflux speed, efflux radius, and kinematic viscosity, the experiments were conducted. Kinematic viscosity was controlled by mixing the glycerin into the water. The table of dynamic viscosity of water-glycerin solution[9] was used and by measuring the density of solution for each concentration, the kinematic viscosity was obtained. With same conditions, same experiment was conducted for 5 times and the average value was used in analysis.

Results and Discussion

Sample hydraulic jump

One sample hydraulic jump was made, and the water depth along the radius was measured.(See Fig. 6.) Near the radius of 6cm, the jump occurred. And with this experiment, the value of k could be known; 0.51.

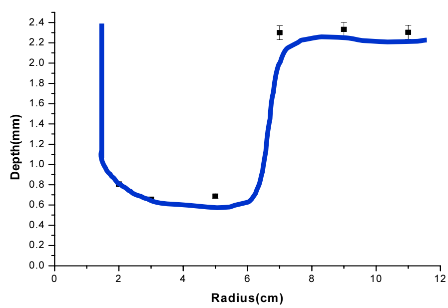

Fig. 6. Sample hydraulic jump

Changes in efflux speed

Fig. 7. is the graph of jump radius as the efflux speed changes. With the equation (13), k value 0.51 and the

condition $a = 2mm$, $\nu = 1\times10^{-6} m^2/s$, theoretically expected red line was made. First five dots are well matched with the line. However, when the speed becomes bigger than critical speed, about $1.6m/s^2$, the radius becomes much bigger than expected. This can be explained by the edge effect of the board. The all equations were made with the assumption that the jump was made on the infinitely wide plate. However, when the board is finite, in our experiment $50cm\times50cm$, the edge effect occurs. When water falls down to the water container, it drags other water of the plate by the cohesion of water. Therefore, the h_2 becomes lower and r_j becomes bigger than expected.

In Fig. 8.(h_1) and 9.(h_2) the expected trend of h_1 and h_2 was identified; as efflux speed increases, h_1 decreases and h_2 increases. The last three dots are also by edge-effect.

Changes in efflux radius

Fig. 10. is the graph of jump radius as the efflux radius changes. With the equation (13), k value 0.51 and the condition $v = 1.4m/s$, $\nu = 1\times10^{-6} m^2/s$, theoretically expected red line was made. We can see the red line is very well matched with the experimental data.

In Fig. 11.(h_1) the expected trend was identified. And in Fig. 12.(h_2), it was identified that over the critical efflux radius, the h_2 remains same value. This can be also explained by the edge effect.

Changes in kinematic viscosity

Fig. 13. is the graph of jump radius as the kinematic viscosity changes with the condition $v = 1.4m/s$, and $a = 1.328mm$. The blue line is experimentally made. With the theoretically expected line, the power of kinematic viscosity was rather different. However, the trend that the jump radius decreases as the kinematic viscosity increases.

Also in Fig. 14.(h_1) and in Fig. 15.(h_2), expected trend was identified; as kinematic viscosity increase, h_1 and h_2 increases both. In Fig. 16., the different shape of hydraulic jump can be identified with the eye between water and water-glycerin.

Conclusion

First, we described what the hydraulic jump is. The overlapping of wave at the critical point was suggested as the reason of the formation of jump. For quantitative investigation, some equations were made with two useful constancy; volume and momentum. With the laminar boundary layer flow, the viscosity influences the jump. At the experimental parts, all the trend of the jump radius and the depth before and after the jump were identified. Especially for jump radius with the variation of efflux speed and radius, the theoretically expected equation was almost perfectly matched with experimental data.

Acknowledgement

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Captions 7-16

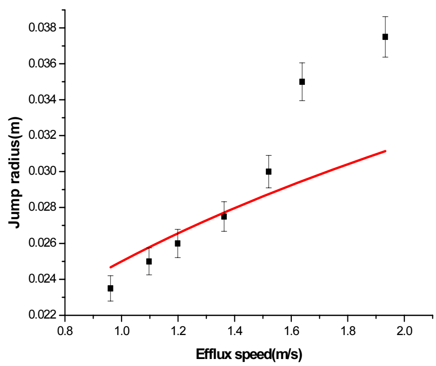


Fig. 7. Jump radius along efflux speed
The conditions are
$$a = 2mm, \nu = 1\times10^{-6} m^2 / s$$

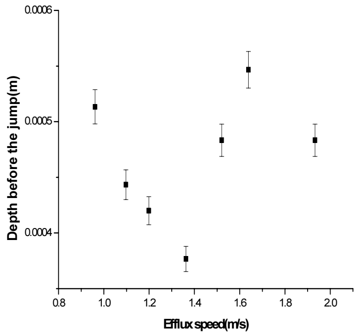


Fig. 8. Changes of h_1 along the efflux speed.
The conditions are $a = 2mm$, and $\nu = 1\times10^{-6} m^2/s$.

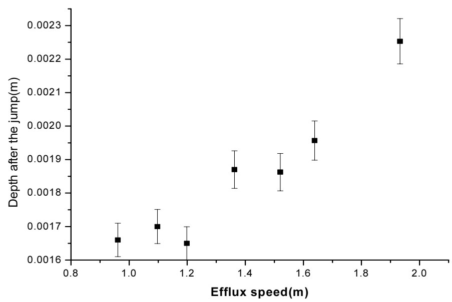


Fig. 9. Changes of h_2 along the efflux speed.
The conditions are $a = 2mm$, and $\nu = 1\times10^{-6} m^2/s$.

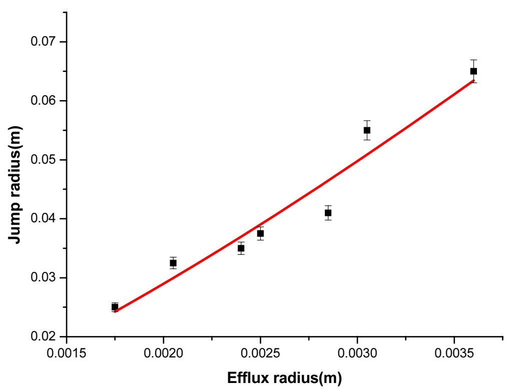


Fig. 10. Jump radius along efflux radius.
The conditions are $v = 1.4 m/s$, and $\nu = 1\times10^{-6} m^2/s$.

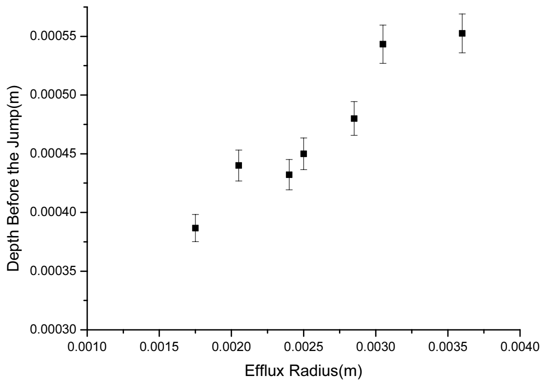


Fig. 11. Changes of h_1 along the efflux radius.
The conditions are $v = 1.4 m/s$, and $\nu = 1\times10^{-6} m^2/s$.

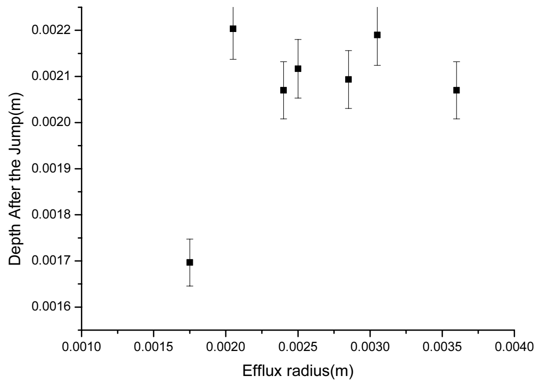


Fig. 12. Changes of h_2 along the efflux radius.
The conditions are $v = 1.4 m/s$, and $\nu = 1\times10^{-6} m^2/s$.

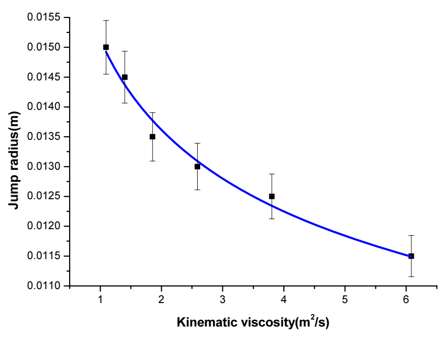


Fig. 13. Jump radius along the kinematic viscosity.
The conditions are $v = 1.4 m/s$, and $a = 1.328 mm$.

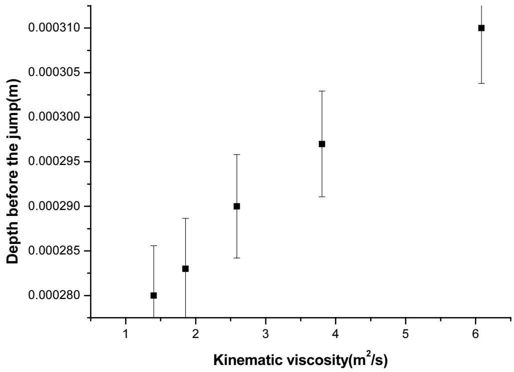


Fig. 14. Changes of h_1 along the kinematic viscosity. The conditions are $v = 1.4 m/s$, and $a = 1.328mm$.

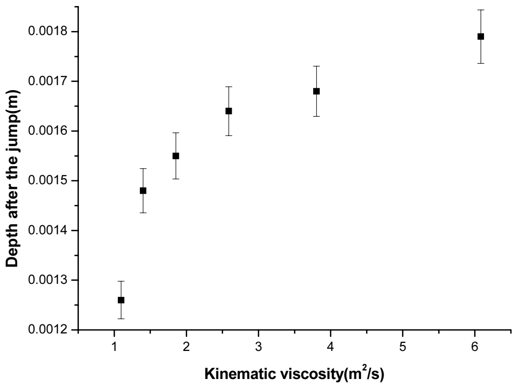
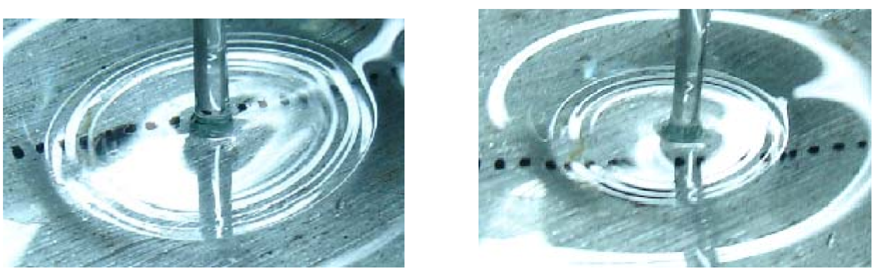


Fig. 15. Changes of h_2 along the kinematic viscosity. The conditions are $v = 1.4 m/s$, and $a = 1.328mm$.



(a) b)

Fig. 16. Hydraulic jump of (a) water, (b) water-glycerin solution