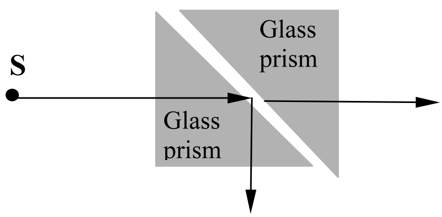
Problem № 15: Optical tunneling

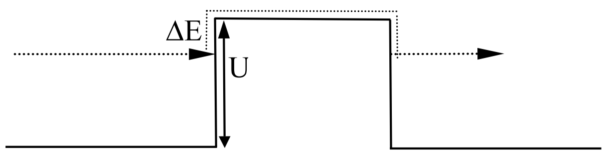
Oleg Matveichuk, Lena Filatova,Grygoriy Fuchedzhy,Alyeksyey Kunitskiy, Valentin Munit
Richelieu lycium, Shepkina Str 5, Odessa, Ukraine

The problem:

Take two glass prisms separated by a small gap. Investigate under what conditions light incident at angles greater than the critical angle is not totally reflected.



At first we should understand what is tunneling. Tunneling is the phenomenon when a particle gets through the potential barrier if its energy is less than barrier's height.



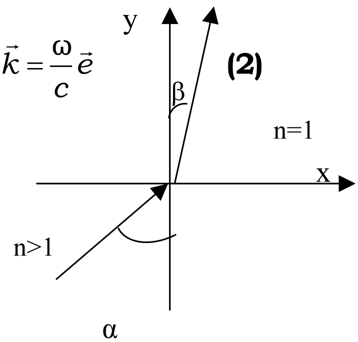
We will investigate this phenomenon from two different sides of view: optical and quantum mechanical.

Let's start with the optical approximation. The electromagnetic wave is traveling in the first prism. Than it comes into the air and starts to dampen. If there is no other prism then it disappears very fast but in our case it continues to go in the second glass prism and we see two rays: reflected and transmitted.

$$E(\vec{r},t) = E_0 \cdot \exp\left(i(\omega t - \vec{k} \cdot \vec{r})\right) \quad \textbf{(1)}$$

At first, lets write the equation of the running wave:
where k is the wave vector of the ray, r is the radius vector of the point at considered moment of time.

$$\vec{k} = \frac{\omega}{c}\vec{e} \qquad \textbf{(2)}$$



After the refraction of the light ray we get:

$$\vec{e} = (\sin\beta; \cos\beta) \qquad \textbf{(3)}$$

$$\vec{k}\cdot\vec{r} = \frac{\omega}{c}\left(x\sin\beta + y\cos\beta\right) \qquad \textbf{(4)}$$

By the Snellius law of refraction:

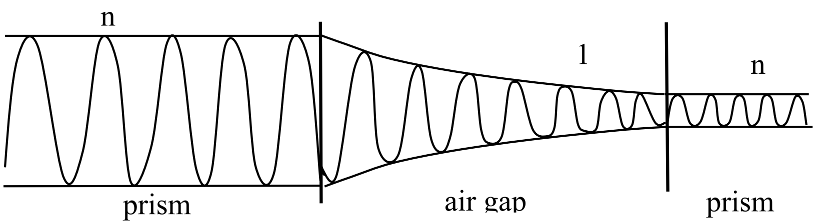
$$\frac{\sin\beta}{\sin\alpha} = n \qquad \textbf{(5)}$$

Condition for the critical angle:

$$n\sin\alpha > 1 \qquad \textbf{(6)}$$

$$E(\vec{r},t) = E_0 \exp\left(-\frac{\omega}{c} y\sqrt{n^2\sin^2\alpha - 1}\right)\exp i\left(\omega t - \frac{\omega}{c} x\sin\alpha\right) \qquad \textbf{(8)}$$

A little manipulation yields:

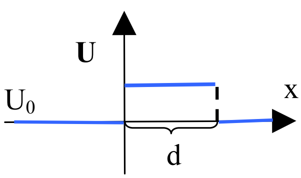


$$\frac{\omega}{c} L_{typical}\sqrt{n^2\sin^2\alpha - 1} = 1 \qquad \textbf{(9)}$$

$$\frac{\omega}{c} = \frac{2\pi}{\lambda}a \quad \Longrightarrow \quad L_{typical} = \frac{\lambda}{2\pi\sqrt{n^2\sin^2\alpha - 1}} \qquad \textbf{(10)}$$

It is obvious that for n$\sin\alpha<1$ light will just reflect and we won't see transmitted ray.

Now we come to the quantum mechanical solution. U(x) is the function of the energy distribution, d is the length of the gap between two prisms.



$$\mathbf{U}(\mathbf{x}) = \begin{cases} 0, \mathbf{x} > 0 \\ \mathbf{U}_0, \mathbf{x} \in [0; \mathbf{d}] \\ 0, \mathbf{x} > \mathbf{d} \end{cases}$$

Let's write Schrödinger's equation:

$$i\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2m}\Delta\Psi + U\Psi \qquad \textbf{(1)}$$

But in our case U doesn't depend on time so we are going to look for the stationary solution:
where

$$\Longrightarrow \qquad \Psi(x,t) = a(x)\exp(-i\omega t) \qquad \textbf{(2)}$$

$$\omega = \frac{E}{\hbar} \qquad \Psi(x,t) = a(x)\exp\left(-\frac{iEt}{\hbar}\right) \qquad \textbf{(3)}$$

$$a(x) = A\exp\left(\pm\sqrt{\frac{2m(U-E)}{\hbar}}x\right) \qquad \textbf{(4)}$$

$$\mathbf{U(x)} = \begin{cases} 0, \mathbf{x} > 0 \\ \mathbf{U_0}, \mathbf{x} \in [0;\mathbf{d}]\mathbf{a} \\ 0, \mathbf{x} > \mathbf{d} \end{cases}$$

$$a(x) = A_1\exp\left(-\sqrt{\frac{2mE}{\hbar}}x\right) \qquad \textbf{(5)}$$

$$a(x) = A_2\exp\left(-\sqrt{\frac{2m(U-E)}{\hbar}}x\right) \qquad \textbf{(6)}$$

$$a(x) = A_3\exp\left(-\sqrt{\frac{2mE}{\hbar}}x\right) \qquad \textbf{(7)}$$

$$A_1 = A_2 \qquad \textbf{(8)}$$

$$A_3 = A_2\exp\left(-\frac{\sqrt{2m(U-E)}}{\hbar}d\right)\exp\left(i\frac{\sqrt{2mE}}{\hbar}d\right) \qquad (9)$$

$$\frac{\sqrt{2m(U-E)}}{\hbar}L_{typical} = 1 \qquad \textbf{(10)}$$

$$L_{typical} = \frac{\hbar}{\sqrt{2m(U-E)}} \qquad \textbf{(11)}$$

Now, when the problem is solved from two points of view we can bring to confrontation optical relevant parameters with mechanical.

$$L_{typical} = \frac{\lambda}{2\pi\sqrt{n^2 \sin^2 \alpha - 1}} \qquad \Longleftrightarrow \qquad L_{typical} = \frac{\hbar}{\sqrt{2m(U-E)}}$$

$$m = \frac{E}{c^2} = \frac{h\nu}{c^2} \qquad c = \lambda\nu \qquad h = 2\pi\hbar$$

$$L_{typical} = \frac{\lambda}{2\pi\sqrt{n^2 \sin^2 \alpha - 1}} \qquad \Longleftrightarrow \qquad L_{typical} = \frac{\lambda}{2\pi\sqrt{2(\frac{U}{E} - 1)}}$$

$$\boxed{n^2 \sin^2 \alpha \qquad \Longleftrightarrow \qquad \frac{2U}{E} - 1}$$

Maxvel's equations have the same mathematical sense as the stationary Schrödinger's equation so their solutions for the corresponding initial and final conditions have to be the same from the mathematical approach. In this problem we see such similarity between relative index of refraction with angle of incident ray and ratio of potential energy of the barrier and of the particle.

Acknowledgments:

Special thanks to:
Oleg Matveichuk, main author of the idea.