

3. Problem •2: Duck`s cone

Solution of Korea

Problem •3: Duck`s cone

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The Problem:

If one looks at the wave pattern produced by a duck paddling across a pond, this reminds one of Mach`s cone. On what parameters does the pattern depend?

The beautiful and fascinating pattern produced behind a moving duck in water has long been studied. Well known as Kelvin Ship Waves, these patterns comprise two components, the diverging waves that form the arms of the wake at a fixed angle of 19.5 degrees and the transverse wake that follows behind a duck. Resembling the Mach`s Cone, the entire pattern follows from the fact that water waves are dispersive, and the group velocity is half the phase velocity in deep water. However, in real situations, non-Kelvin wave patterns are also observed too. Experiments were performed to identify the parameters that determine the duck`s cone.

Keywords

Duck`s cone, Kelvin Ship Waves, group velocity, dispersion, deep water

I. Introduction

The cone-shaped pattern left behind the trails of a duck in water has fascinated the minds of people for a long-time. Undistinguishable from the pattern produced by a ship, the pattern is well known as Kelvin Ship Waves, named after Lord Kelvin who first gave a theoretical explanation for this phenomenon. The Kelvin Ship Wave consists of two components, the diverging waves and the transverse wake. Made of two wake lines that together form the “V”, the diverging waves look feathery in appearance and move stationary relative to the boat. Each arms of the V makes an angle of about 19.5 degrees with respect to the trajectory of the boat.

The second component, the transverse waves, fills the area between the arms of the V. They resemble arcs of a circle, and also move stationary to the boat. However, the radius of curvature and the shape does not stay constant when the velocity of the boat changes, and even seem to even disappear at high speeds.

The pattern follows from the fact that water waves are dispersive, and the group velocity is half the phase velocity in deep water. The first derivation of this pattern was given by Lord Kelvin, and the theory of water waves need to be studied in advance to understand this

beautiful phenomenon. Resembling the Mach's Cone, the duck's cone is a much more complex phenomenon determined by the characteristics of water waves.

II. Theory

We first investigate the case of a point source of wave moving through a nondispersive wave at supersonic speed. As in the case of a bullet in air, all waves have the same phase and group velocity in a non-dispersive medium. The angle of the cone generated by the body in supersonic movement, called the Mach angle, is totally dependent on the relative velocity of the body and the waves produced. That is, the angle θ between the wavefront, one arm of the V, and the velocity direction of the moving point source is given by following Equation .

$$v_\theta = v_0 \sin \phi$$

Dispersion relation relates the angular frequency ω to the wave number k . The derivation of the dispersion relation requires the application of Bernoulli's theorem, which states that the total energy per unit mass has the same value at each point along a given streamline. The dispersion relation for water waves is

$$\omega^2 = (gk + \sigma k^3 / \rho) \tan kh$$

The first term, gk , is the gravity term and the second is the surface tension term. The surface tension term is significant for short wavelengths, which we call ripples. When the two terms in the bracket are equal, at 20°C, $\sigma = 0.073 \text{ N/m}$, for water, and $\rho = 1000 \text{ kgm}^{-3}$, the critical wavelength is 17mm. Waves much shorter will be dominated by surface tension. Neglecting the gravity term, and assuming $kh \gg 1$ since the water is assumed to be deep, $\tan kh \approx 1$, and we have the equation that describes the short waves dominated by surface tension.

$$\omega = (\sigma k^3 / \rho)^{1/2}$$

However, waves that are larger than 17mm are gravity waves. Recalling the dispersion relation, $kh \gg 1$ in deep water, so $\tan kh \approx 1$. Surface tension is negligible and the dispersion relationship reduces to

$$\omega = (gk)^{1/2}$$

The former equation shows the small ripples created in deep water while later equation shows the long gravity waves, that we usually observe, created in deep water.

While phase velocity refers to the velocity at which the signal is propagating through the media, group velocity refers to the velocity at which the actual energy is transmitted. In case of waves with small internal oscillations and the wave envelope surrounding the traveling wave, phase and group velocity may differ. The phase velocity of ripples in deep water and group velocity of ripples in deep water are described as follows.

$$v_p = \omega / k \approx (\sigma k / \rho)^{1/2} = (2\pi\sigma / \rho\lambda)^{1/2}$$

$$v_g = d\omega / dk \approx \frac{3}{2} (\sigma k / \rho)^{1/2} = \frac{3}{2} v_p$$

We see that the shortest waves travel the fastest in ripples. Also, because phase velocity is lower than group velocity, individual ripples appear to travel backwards the parent group as it propagates. In case of gravity waves in deep water, the phase velocity and group velocity of gravity waves in deep water are described as follows. We see that the longest waves travel the

fastest, unlike ripples, in gravity waves. Interestingly, the group velocity of gravity waves in deep water is half the phase velocity.

$$v_g = d\omega / dk \approx \frac{1}{2}(g/k)^{1/2} = \frac{1}{2}v_p$$

$$v_p = \omega / k \approx (g/k)^{1/2} = (g\lambda/2\pi)^{1/2}$$

Sea-shore waves being the typical example, shallow water waves are approximately non-dispersive. Neglecting surface tension and assuming that $kh \ll 1$, $\tan kh \approx kh$, and solving the equation for the velocity of water waves we obtain and see that all waves travel at approximately same velocity in shallow water, and shallow water waves are thus non-dispersive.

$$v_p \approx v_g \approx \sqrt{gh}$$

We start with the assumption that a traveling duck of speed v_0 is a point source of gravity waves having a broad wavelength spectrum. First, we consider a narrow band of wavelengths centered around a given wavelength λ , traveling with a phase velocity, v_p . Suppose the duck travels from B1 to B2 as shown in Figure1. During the time interval t , the duck travels from B1 to B2. The phase velocity of the given wave would reach W, and the wave front produced would form an angle ϕ . However, because of the interference of the narrow band of wavelengths, the wave crest would travel at the group velocity to the midpoint between B1 and B2 during time t , as shown in Figure 2.

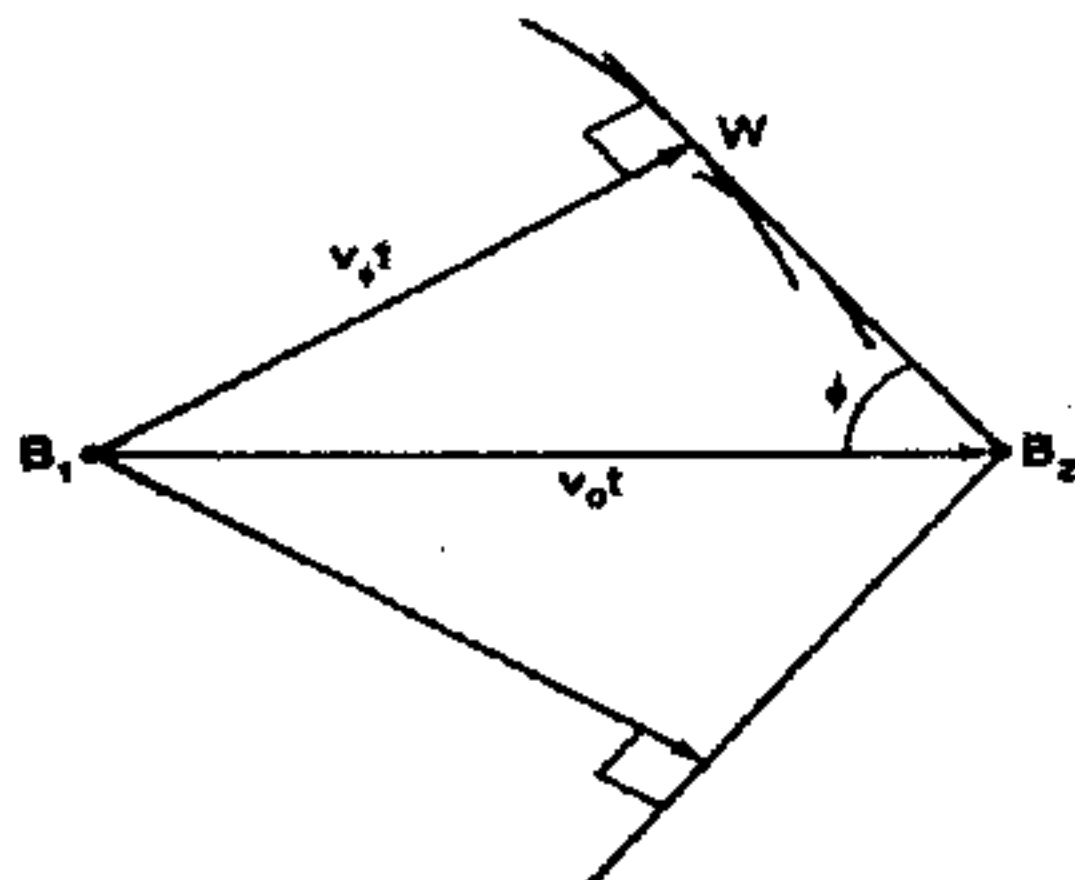


Figure1.

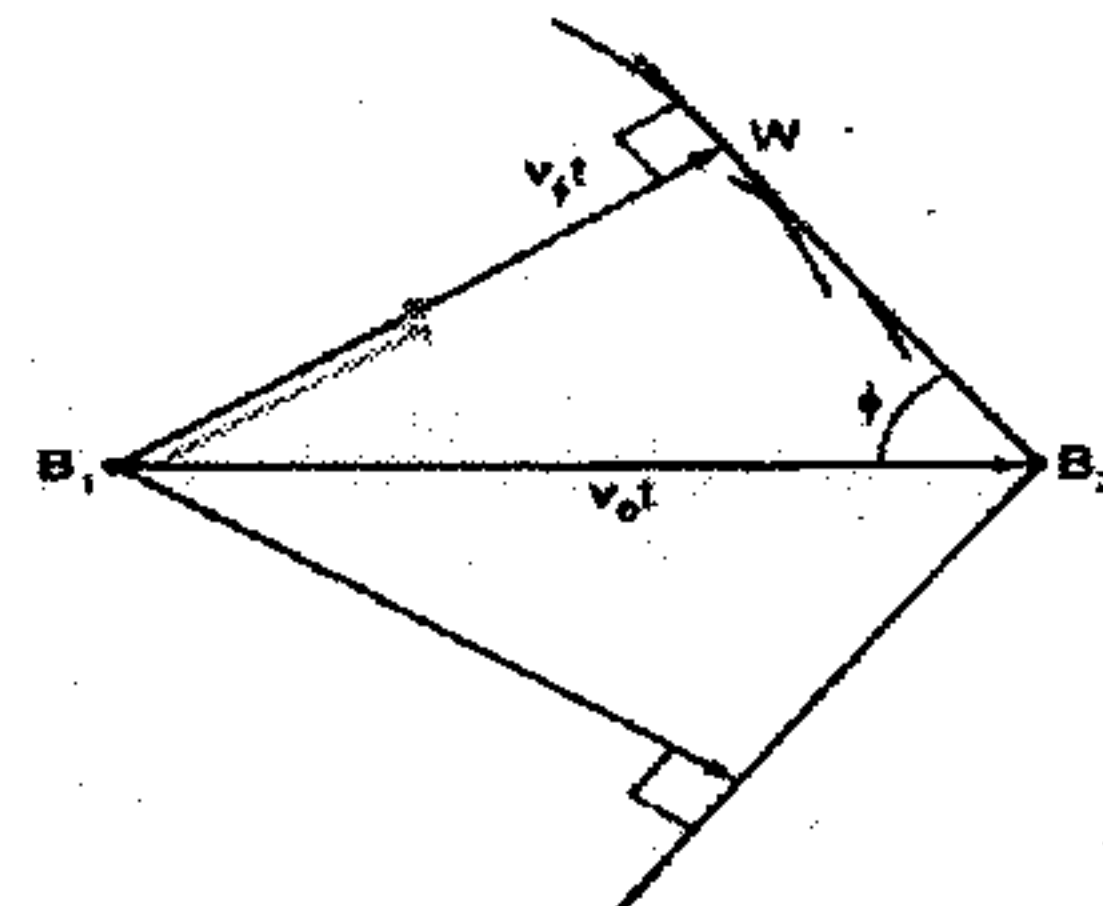


Figure2.

Thus, for the given narrow band of wavelengths centered around λ we would observe the constructive interferences at the midpoints from B1 to B2, as shown in Figure3.

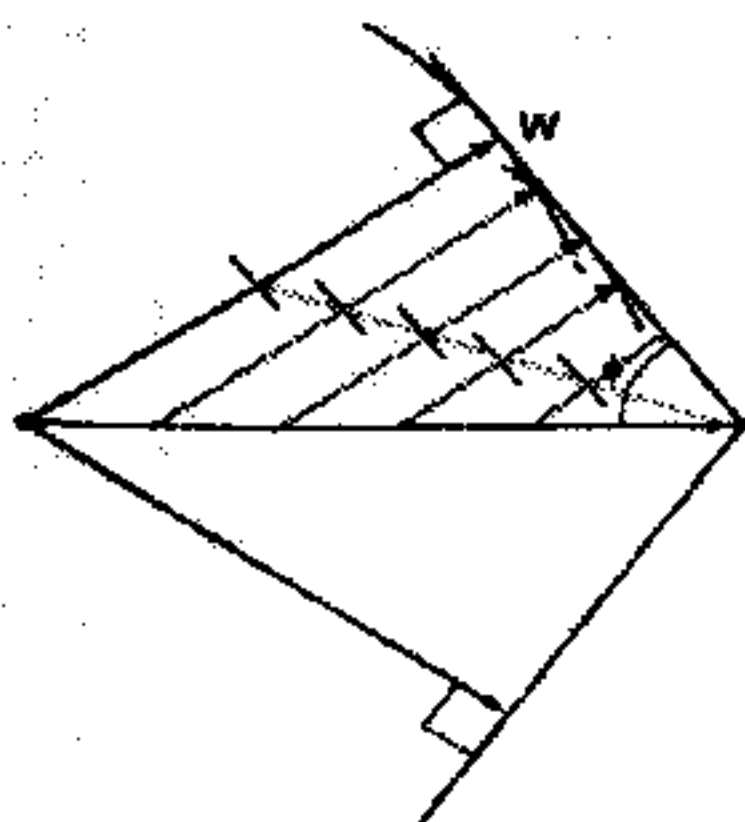


Figure3.

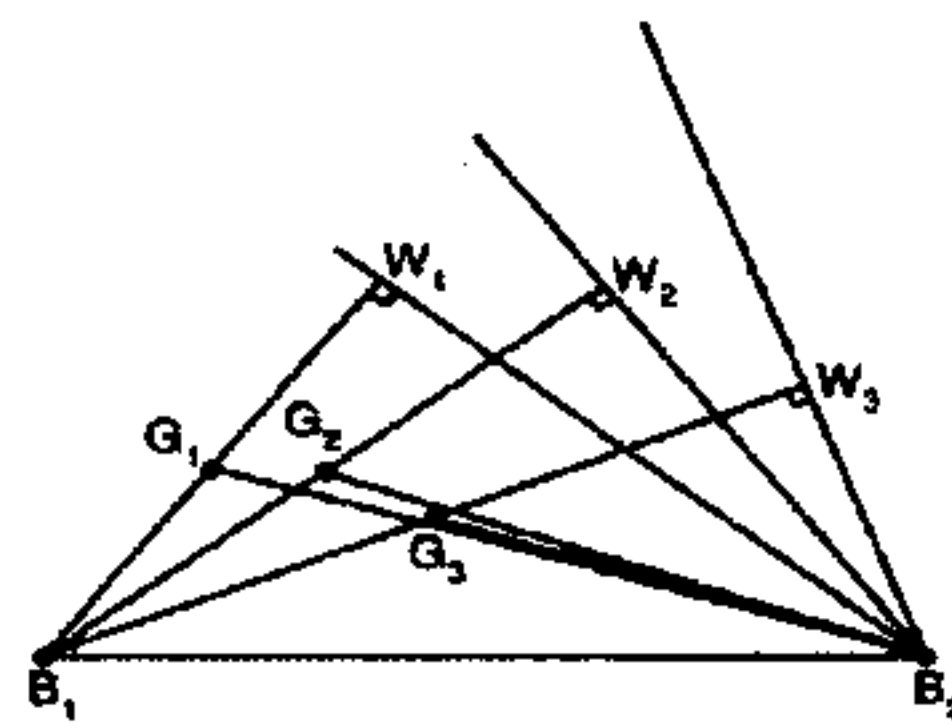


Figure4.

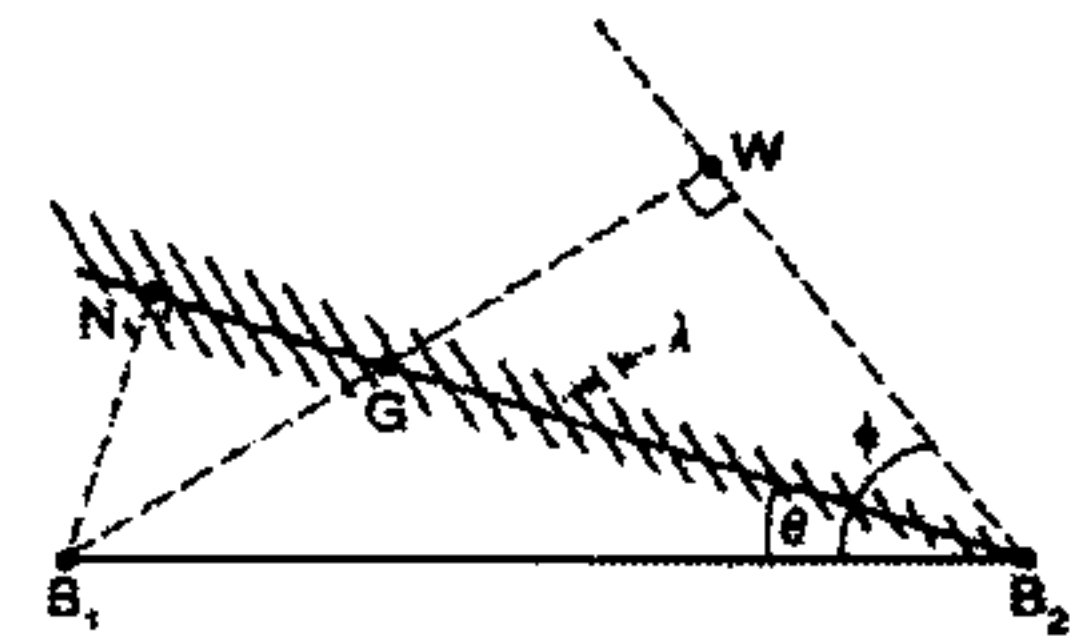
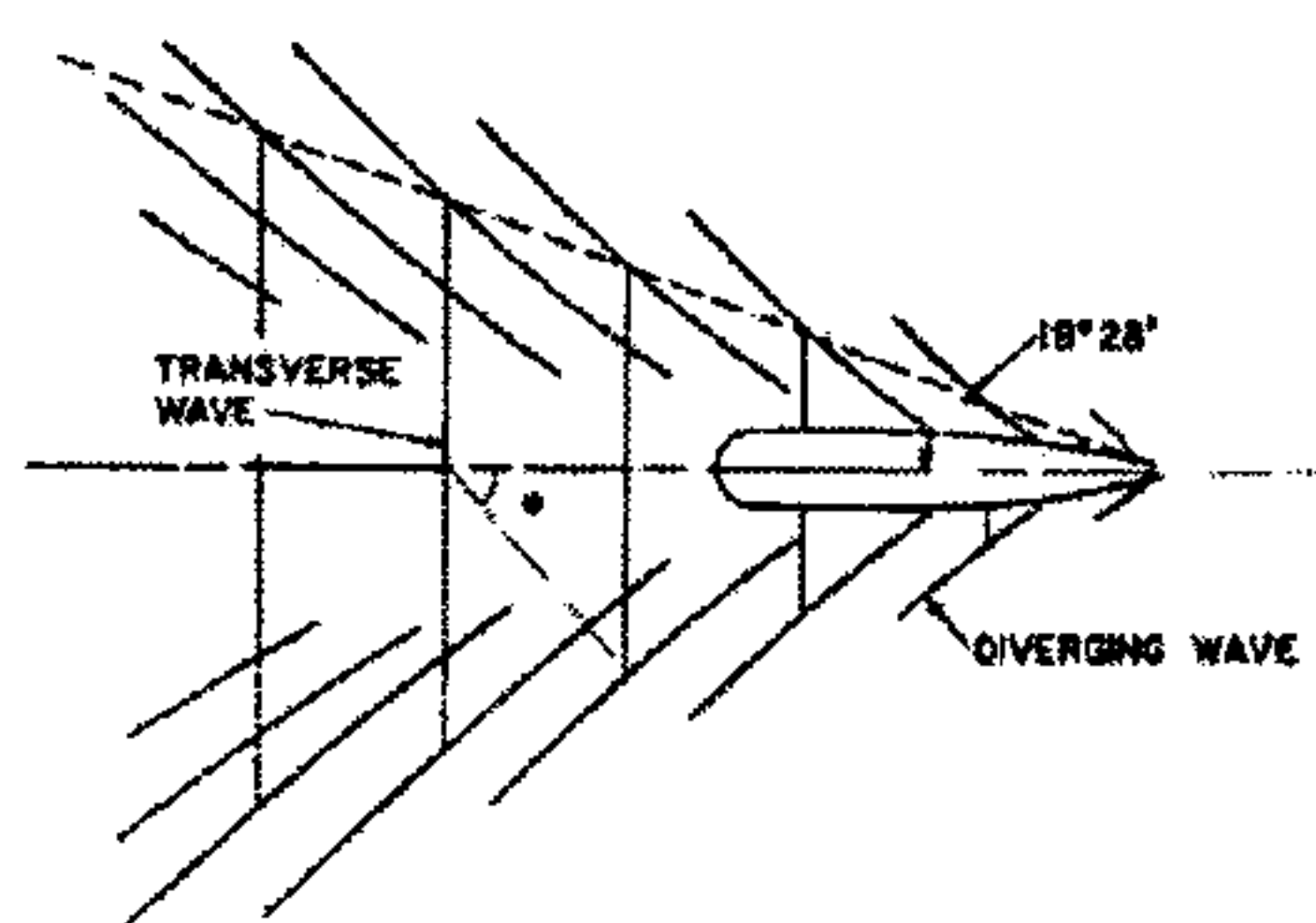


Figure5.

However, the duck produces a broad wavelength spectrum. Deep water gravity waves, as mentioned before, are dispersive waves, and waves of different wavelengths travel different distances during the time interval. That is, as shown in Figure 4, different waves arrive at different points of W. Geometrically, we could see that the maximum angle ϕ produced by the waves is 19.5 degrees. Choosing units in Figure 5, such that $B2W = 1$, $B1G = GW = a$, So, we see that the maximum angle is about 19.5 degrees, explaining the fixed angle of 19.5 degrees observed in Kelvin waves.

Because the wave is fixed relative to the duck, the phase velocity where the waves cross the boat trajectory is same as the boat speed. Thus, the waves we observe are centered around the wave having the phase velocity of v_0 . Thus as the boat moves from B1 to B2, the phase velocity of the waves would have propagated to B2. However, because of group velocity and interference, we would observe the wave crest at halfway between B1 and B2. Thus, we would expect that for a transverse curved wave we observe at a distance L behind the boat, it was generated at a distance 2L behind the boat. The radius of curvature would be L.



VI. Experiment

A cart was used to move objects through a transparent water tank. The speed of the cart was controlled, with a DC power supply. The speed of the cart was measured with PASCO photogates. To clearly observe the pattern created by the waves, the shadows of the waves were observed. Light was shone from above the tank. A white paper covered the lower surface of the water tank. The shadows on the paper were captured with a camera.

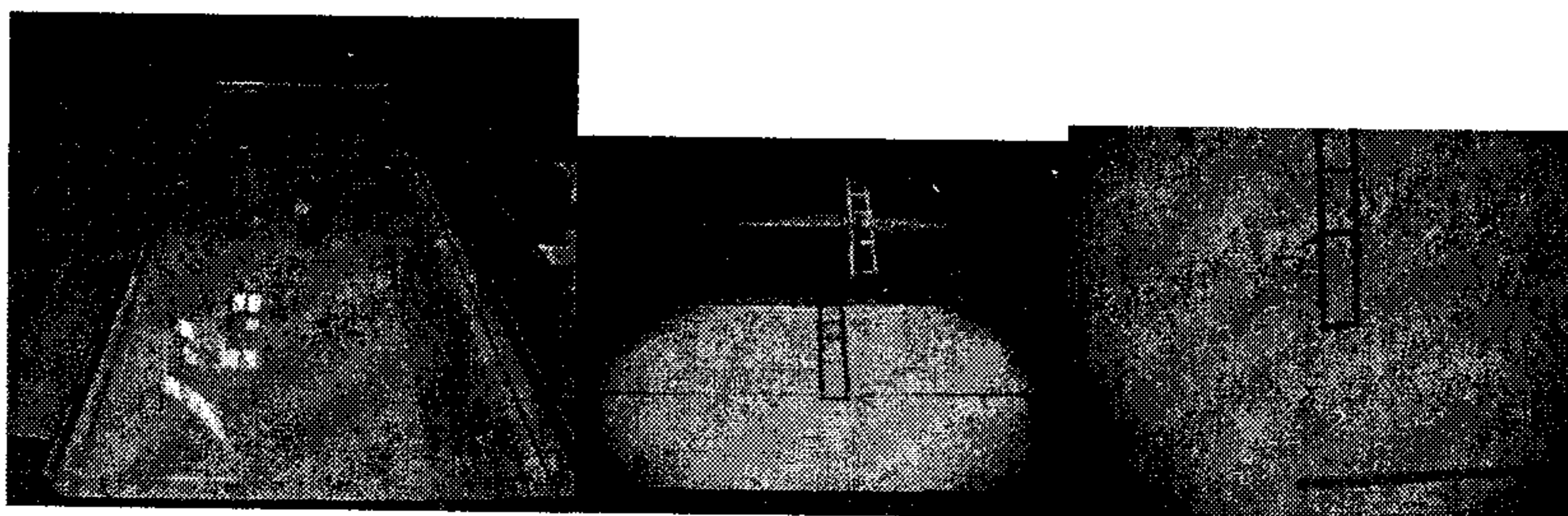


Figure 7

First, the angle of the 'V' diverging waves was measured at different speeds in deep water, which was over 50cm-water height. Next, angles of diverging waves were measured at

different depths to see the effect of water depth on the duck's cone. Next, the length of the diverging waves, the wavelength, was measured. Then, the transverse waves were observed. Finally, the experiment was done with different shape of the object.

VII. Experiment Results

1. Angle of Diverging of Waves at different Speeds



Figure8.

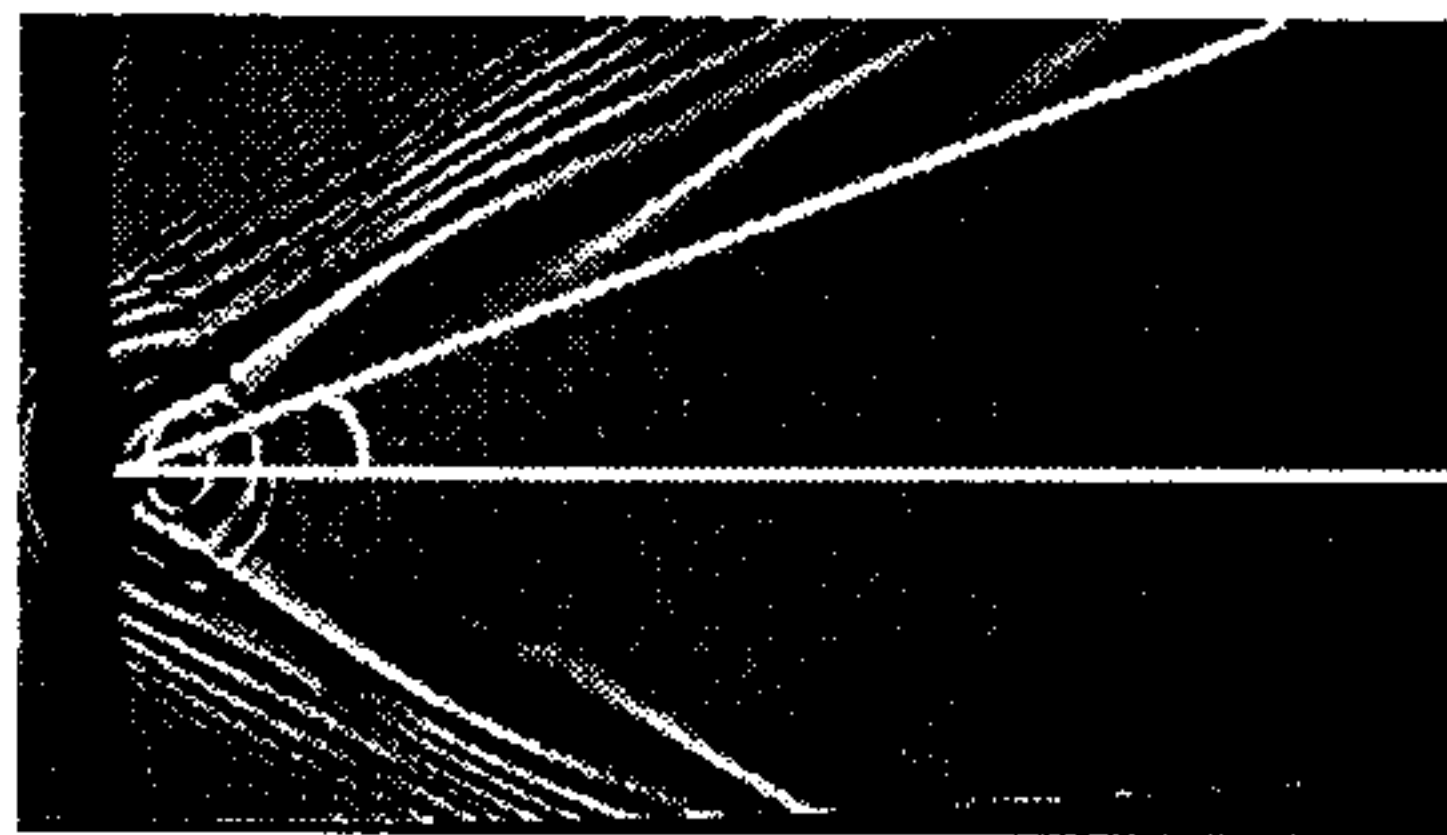


Figure9.

The wave pattern shows changes as shown in figure(8) as the speed increases. The images were analyzed as shown in figure 9, connecting points where the diverging waves at the boundary between the transverse and diverging waves.

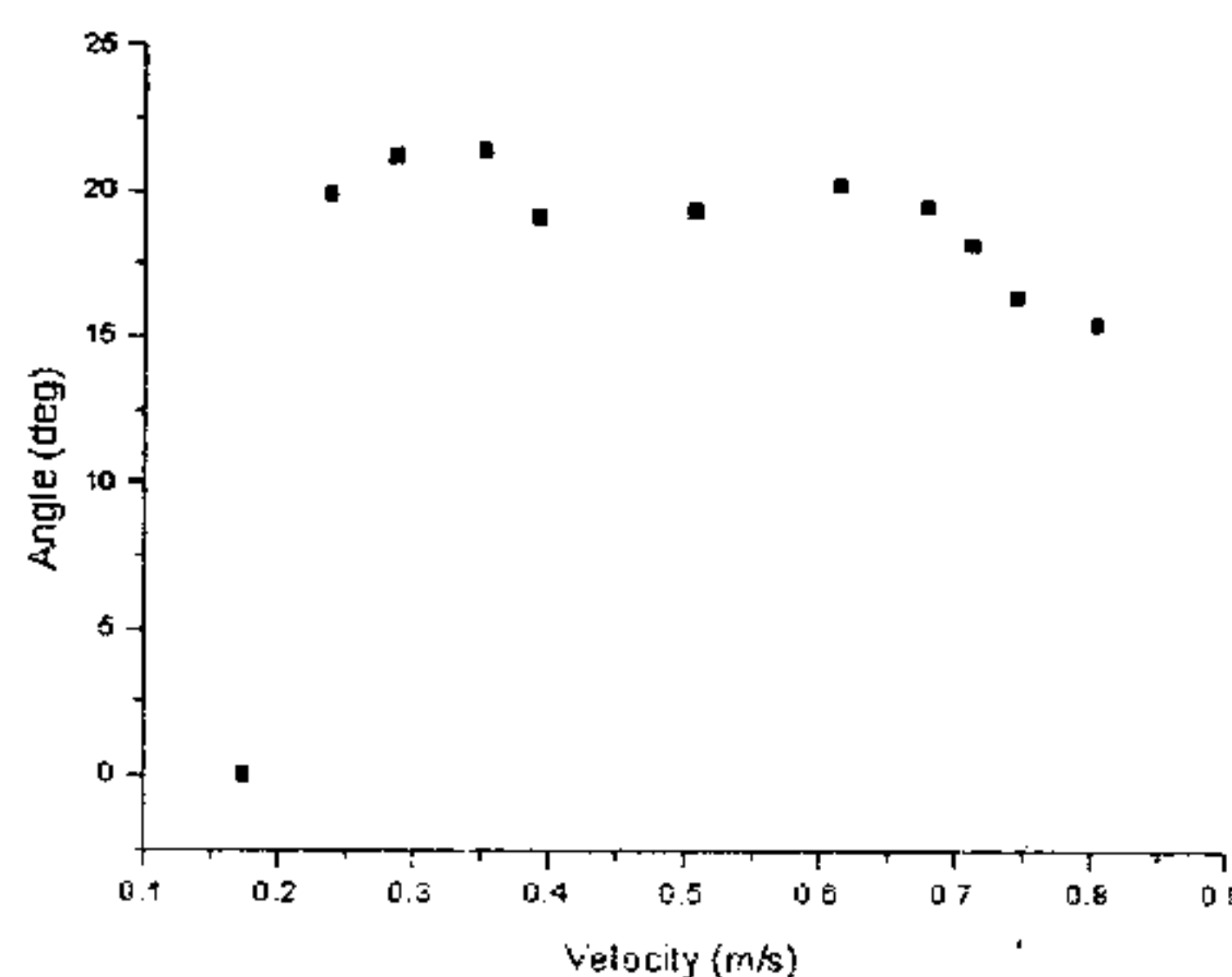


Figure10.

Interestingly, there was a limit speed that produces the V waves at about 0.2 m/s. The dot below shows that no values exist. Also, the angle slowly decreased below 19 degrees as speed increased. This may possibly be caused by the fact that the water height was not actually

'deep' in the sense that it was not infinitely deep. The limit speed can be explained by the fact that water waves require a minimum amount of energy to be produced.

Also, capillary waves, which are short-wavelength waves shorter than a 1.7 centimeters were observed just outside the V-diverging waves. The wavelength of the capillary waves does not vary with the different speeds of duck.

2. Transverse Waves

The length of transverse wave increases with velocity as shown in the above figure (11). Seeing that the transverse waves are stationary relative to the duck, the relationship between duck speed and the transverse wavelength can be induced as $\lambda = 2\pi v_0^2 / g$. The length of transverse wave increases with velocity as shown in the above figure (11). Seeing that the transverse waves are stationary relative to the duck, the relationship between duck speed and the transverse wavelength can be induced as

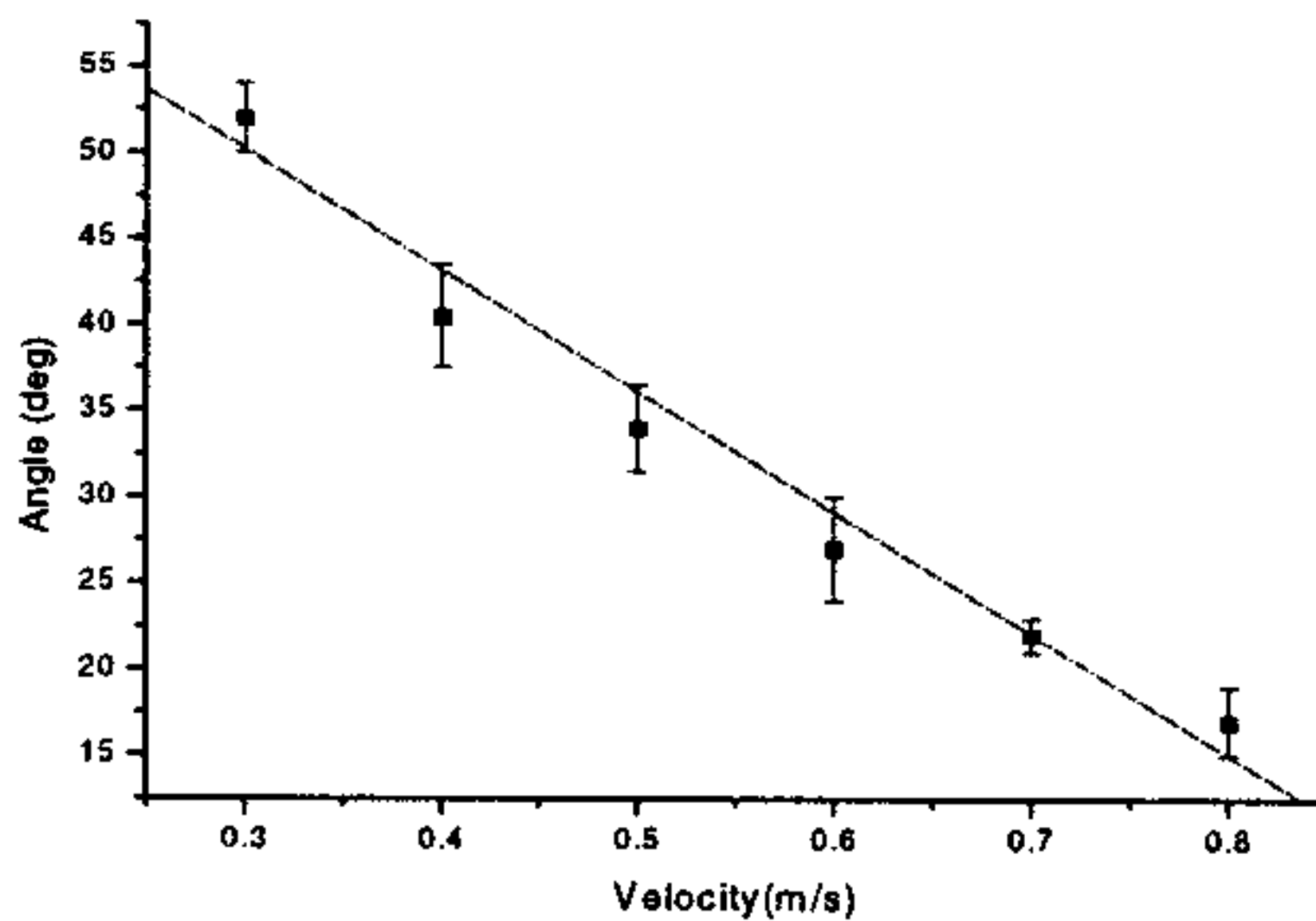


Figure 11.

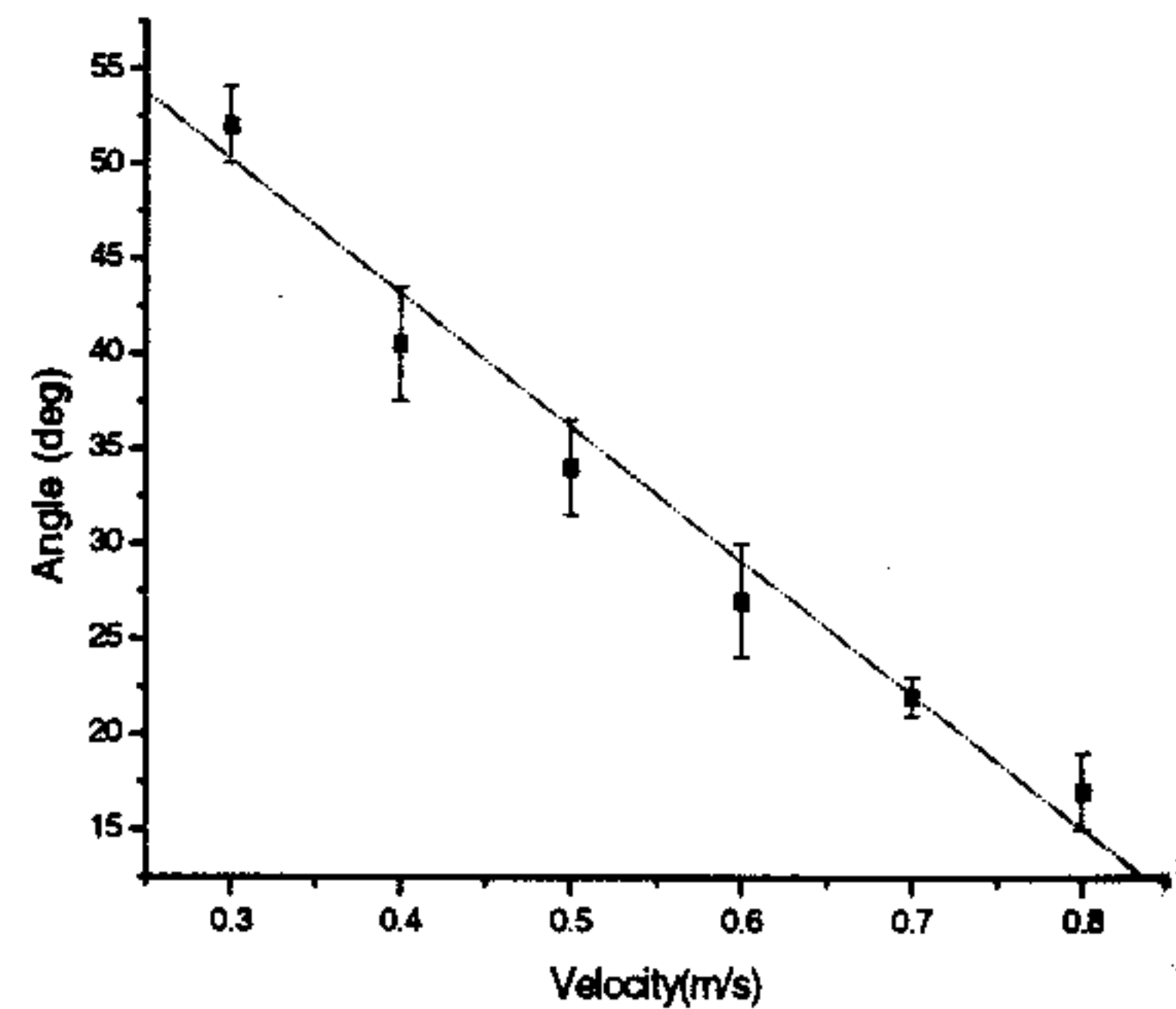


Fig. 12

3. Shallow Water

Angle of diverging waves was measured at different speeds in shallow water. Gravity waves, the typical wave of the Kelvin Ship wave that is over 1.73cm, were not observed. Angle linearly decreases similar to the mach's cone. This can be explained by the fact that water waves are approximately non-dispersive in shallow water, and thus the phase and group velocity do not differ. The angle would depend on the relative speed of the wave and the object.

3. Different Shapes

Four shapes were used. The first shape had a round front and a pointy back. The second was just a reverse of the first one. The third shape was round at all sides, and the last one was pointy both at front and back. Experiment was carried out for different shapes in deep water at different speeds.

Figure (14) shows the results. Each column represents different speeds of 0.3 m/s, 0.5 m/s, and 0.7 m/s respectively. Each row represents the different shapes as mentioned before. At a

low velocity of 0.3 m/s, gravity waves and capillary waves were clearly observed. However, as velocity increases, great amount of bubbles and turbulence dominated the wake. This is probably due to the fact that the quick displacement of water by the shapes caused a swift movement of water which caused bubbles and turbulence.

Also, unlike the previous experiments done with an ideal point, a pin, the wake pattern produced by a distinct shape showed a gap depending on the ship length as shown in Figure (13). Although the wakes produced by 4 differently shaped objects did not show much difference with one another, they all showed huge differences with that of the pin.

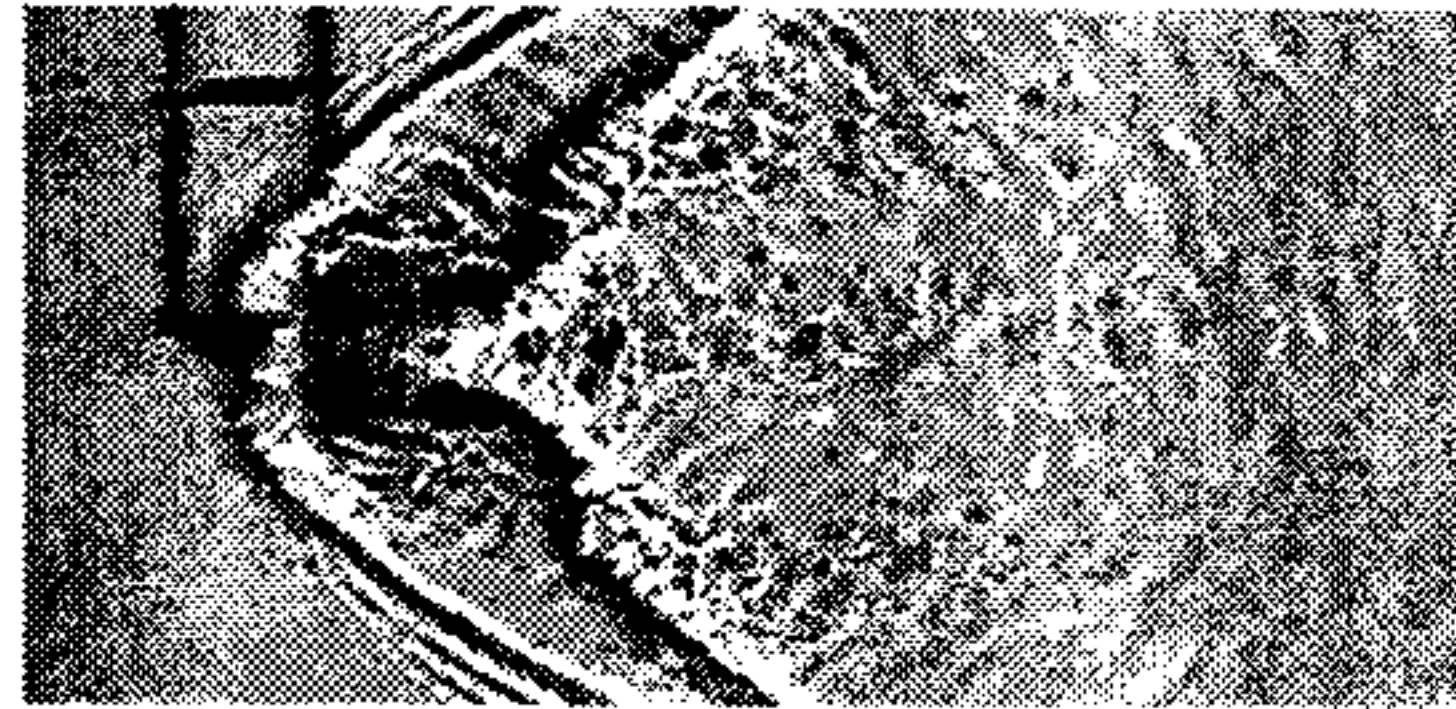


Figure13.

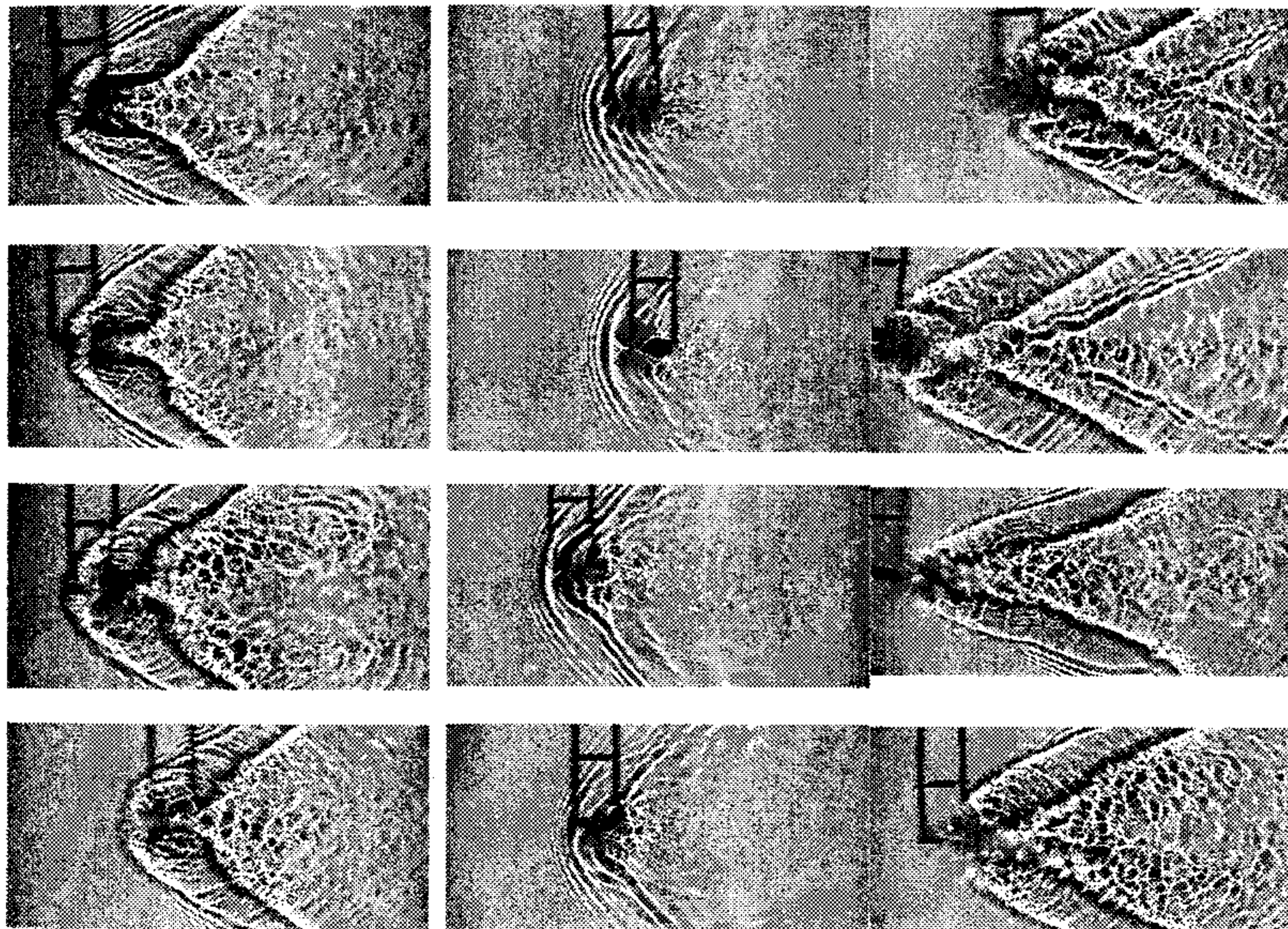


Figure14.

VIII. Conclusion

The pattern observed behind a duck moving in water is known as Kelvin Ship waves. It has two components, the 'V' diverging waves and the transverse waves. Interestingly, the diverging waves form a constant angle of about 19.5 degrees with the trail of the duck. The

entire pattern is constructed from the fact that deep water is dispersive and group velocity is half phase velocity. In case of shallow water, the speed of the duck is a main parameter. Kelvin ship waves are not observed, but simple cone-shaped V follows. In case of deep water, the speed of the duck only changes the wavelength, without changing the overall pattern. Capillary and gravity waves are distinguishable, and capillary waves are observed just outside the diverging pattern. Changing shape does not alter the wake pattern, but turbulence and bubbles makes it highly different from the duck's cone produced with an ideal point.

References

References (*Reference* style) should be listed in alphabetic order by surname at the end of the paper.

Each line of a reference, except the first, should be indented 0.5" from the left margin.

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