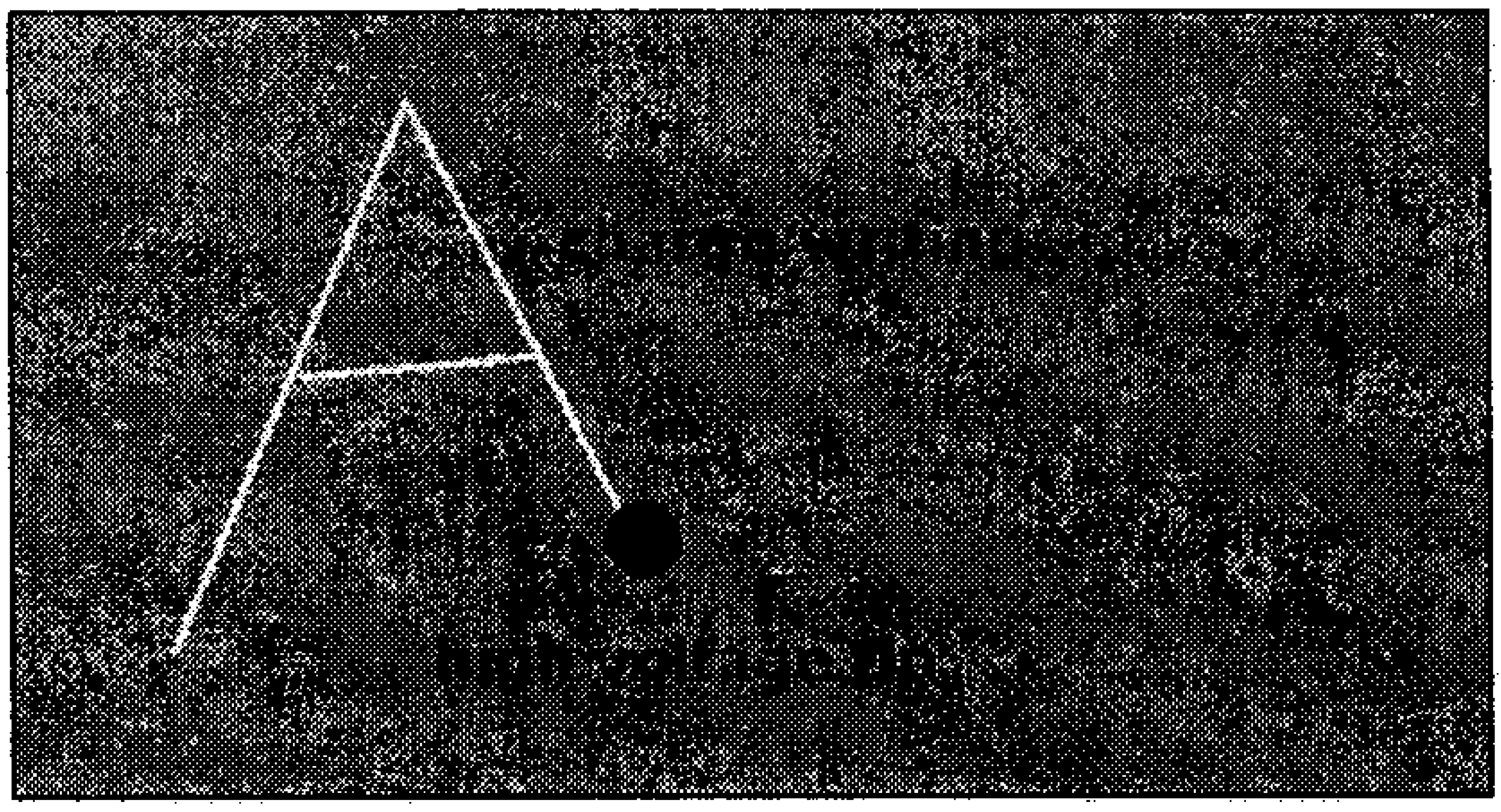

Fig.11

7. Problem №10: Inverted pendulum

7.1. Solution of Korea

Problem №10: Inverted pendulum

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The Problem:
It is possible to stabilize an inverted pendulum It is even possible to stabilize an inverted multiple pendulum /one pendulum the top of the other/. Demonstrate the stabilization and determine on which parameters this depends.

This paper studies comprehensively about the methods of stabilizing an inverted pendulum. An inverted pendulum is a free hung pendulum which is upright, and just like an ordinary pendulum, it naturally falls downward because of gravity. Thus, the inverted pendulum system is inherently unstable. In order to keep it upright, or stabilize the system, one needs to manipulate it, either vertically or horizontally.

Many stabilizing methods have been developed. In 2-Dimesional system, an inverted pendulum can be stabilized thorough either vertical or horizontal oscillation with certain frequency. In 3-Dimension, rotational arms or free robot arms are used for stabilization. For algorithm, a controller using feedback system or simple oscillation both work to keep the pendulum upright, though processes or extents of stability are different from each other.

This paper first proposes theoretical background for all the cases. Then, the experiments focus on horizontal oscillation and delve into the various characteristics and factors of stabilization pattern.

A. Physical Modeling for the Inverted Pendulum

Mechanics of the inverted pendulum is not different from that of the ordinary pendulum. It consists of a rod and a pivot. When you draw a force diagram of inverted pendulum system, it's shown as Figure 1. As seen in the diagram, force is applied to the pendulum's pivot (base).

Let m be the mass of a rod, P(x_0, y_0) coordination of the pivot, CM(x,y) coordination of the center of mass, l a distance from P to CM.

Two basic motion equations for the pendulum system are

$$m\ddot{x} = F_x, \quad m\ddot{y} = F_y - mg \tag{1}$$

and

$$I_c\ddot{\theta} = lF_y \sin\theta - lF_x \cos\theta \tag{2}$$

Where Ic is the moment of inertia of the rod with a pivot on CM.

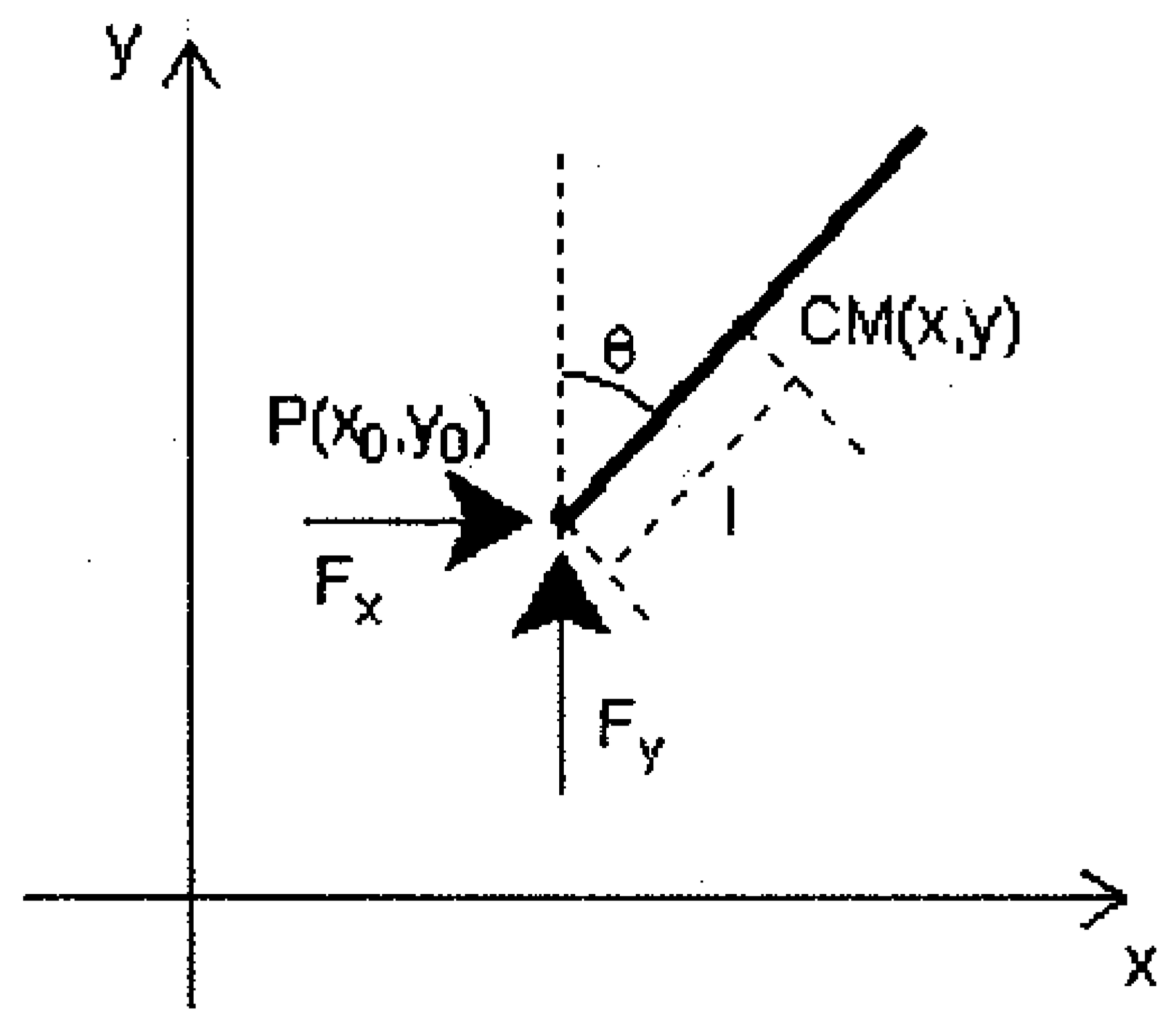


Figure 1 Force Diagram of the Inverted Pendulum

Substitute Equation (1) for Fx and Fy in Equation (2), then we obtain

$$I_c\ddot{\theta} = ml(\ddot{y} + g)\sin\theta - ml\ddot{x}\cos\theta \tag{3}$$

Then, from relationship between the coordinate of P and that of CM,

$$x = x_0 + l\sin\theta, y = y_0 + l\cos\theta \tag{4}$$

Thus,

$$\begin{aligned} \ddot{x} &= \ddot{x}_0 + l\ddot{\theta}\cos\theta - l\dot{\theta}^2\sin\theta, \\ \ddot{y} &= \ddot{y}_0 - l\ddot{\theta}\sin\theta - l\dot{\theta}^2\cos\theta \end{aligned} \tag{5}$$

Substitute Equation (5) for $\ddot{x}$ and $\ddot{y}$ in Equation (3), then the equation becomes

$$\frac{I_p}{mL}\ddot{\theta} + \ddot{x}_0\cos\theta - (\ddot{y}_0 + g)\sin\theta = 0 \tag{6}$$

Where $I_p = I_c + ml^2$ is the moment of inertia of the rod with a pivot on P. Equation (6) is a universal motion equation for the inverted pendulum, which can also be applied to an ordinary pendulum. This equation will be used in simulation which will be later explained. In the next chapter, we will use this equation to find stabilizing methods of the pendulum.

B.Stabilizing Methods of the Inverted Pendulum

An inverted pendulum can be stabilized in mainly two ways: simple oscillation and feedback control system. The pendulum with oscillating base at certain frequency can stay upright without falling down. With a control system, the pendulum's movement is minutely manipulated by the machine to keep it upright.

i) Oscillation

Oscillation is a possible method for stabilizing the inverted pendulum. Three kinds of oscillation are possible : oscillating vertically, oscillating horizontally, and rotating it. (Blitzer 1965)

a) The vertically driven pendulum

The inverted pendulum can be stabilized by moving it up and down at certain frequency. Mathematically,

$$x_0 = 0, \;\; y_0 = A\cos\omega t \tag{7}$$

Then, Equation (6) becomes

$$\frac{I_p}{ml}\ddot{\theta} + (A\omega^2\cos\omega t - g)\sin\theta = 0 \tag{8}$$

b)The pendulum driven in two dimensions

Here, $x_0 = B\cos(\omega' t + \Phi), \;\; y_0 = A\cos\omega t$. Equation (6) becomes

$$\frac{I_p}{ml}\ddot{\theta} - B\omega'^2\cos(\omega' t + \Phi)\cos\theta + (A\omega^2\cos\omega t - g)\sin\theta = 0 \tag{9}$$

c) The rotating pendulum

The rotating pendulum is the special subcase for the case two, where $B = A, \;\; \omega = \omega', \;\; \Phi = -\pi/2$. In other words, $x_0 = A\sin\omega t, \;\; y_0 = A\cos\omega t$. Then, Equation (6) becomes

$$\frac{I_p}{ml}\ddot{\theta} + A\omega^2\sin(\theta - \omega t) - g\sin\theta = 0 \tag{10}$$

d)The pendulum driven horizontally

In this case, $x_0 = A\cos\omega t, \;\; y_0 = 0$

$$\frac{I_p}{ml}\ddot{\theta} - A\omega^2\cos\omega t - g\sin\theta = 0 \tag{11}$$

For small oscillation, $\theta << 1$, the angle range where the inverted pendulum can be stabilized, Equation (11) is simplified into

$$\frac{I_p}{ml}\ddot{\theta} - A\omega^2 \cos\omega t - g\theta = 0 \qquad (12)$$

$$\ddot{\theta} - {\omega_0}^2\theta = D\cos\omega t \qquad (13)$$

where ${\omega_0}^2 = \frac{mgl}{I_p}$, $D = \frac{ml\omega^2 A}{I_p}$

In this equation ${\omega_0}^2$ is remarkable because it is an angular frequency of the normal pendulum. Equation (13) is a linear differential equation, and the solution for this differential equation in terms of theta is

$$\theta = ae^{\omega_0 t} + be^{-\omega_0 t} - \frac{D\cos\omega t}{\omega^2 + {\omega_0}^2} \qquad (14)$$

$$a = \frac{1}{2}[\theta_0 + \frac{\dot{\theta}_0}{\omega_0} + \frac{D}{\omega^2 + {\omega_0}^2}]$$

$$b = \frac{1}{2}[\theta_0 - \frac{\dot{\theta}_0}{\omega_0} + \frac{D}{\omega^2 + {\omega_0}^2}] \qquad (15)$$

where the angular displacement and the angular velocity of the inverted pendulum at t=0 is θ_0 and $\dot{\theta}_0$, respectively.

In order to make the pendulum oscillate, the exponentially increasing term, or the exponential term with a positive exponent, should be eliminated. Thus, a should be equal to zero, or

$$\dot{\theta}_0 = -\omega_0\theta_0 - \omega_0\frac{D}{\omega^2 + {\omega_0}^2} = -E\omega_0 \qquad (16)$$

where $E = \theta_0 + \frac{D}{\omega^2 + {\omega_0}^2}$

Then, the final solution for theta becomes

$$\theta = Ee^{-\omega_0 t} - \frac{D\cos\omega t}{\omega^2 + {\omega_0}^2} \qquad (17)$$

When t approaches to infinity, theta converges to

$$\theta = -\frac{D\cos\omega t}{\omega^2 + {\omega_0}^2} \qquad (18)$$

To conclude, when the pendulum's pivot oscillates in $x_0 = A\cos\omega t,\ y_0 = 0$, it can be stabilized with $\theta = -\frac{D\cos\omega t}{\omega^2 + {\omega_0}^2}$. Although theta does not converge into one value, it does keep upright while oscillating constantly. The experiments, which will be explained later, focus on the relationship between x_0 and theta.

ii) **Feedback Control system.**

Another method to stabilize the inverted pendulum is using feedback control system. Feedback control system is the process in which the movement of the pendulum's pivot is continuously fed back based on the pendulum's physical condition in order to keep the pendulum upright. For example, at one moment, when the pendulum leans to the right, the base moves fast to the right so that the pendulum becomes upright. Still, the base should move in a

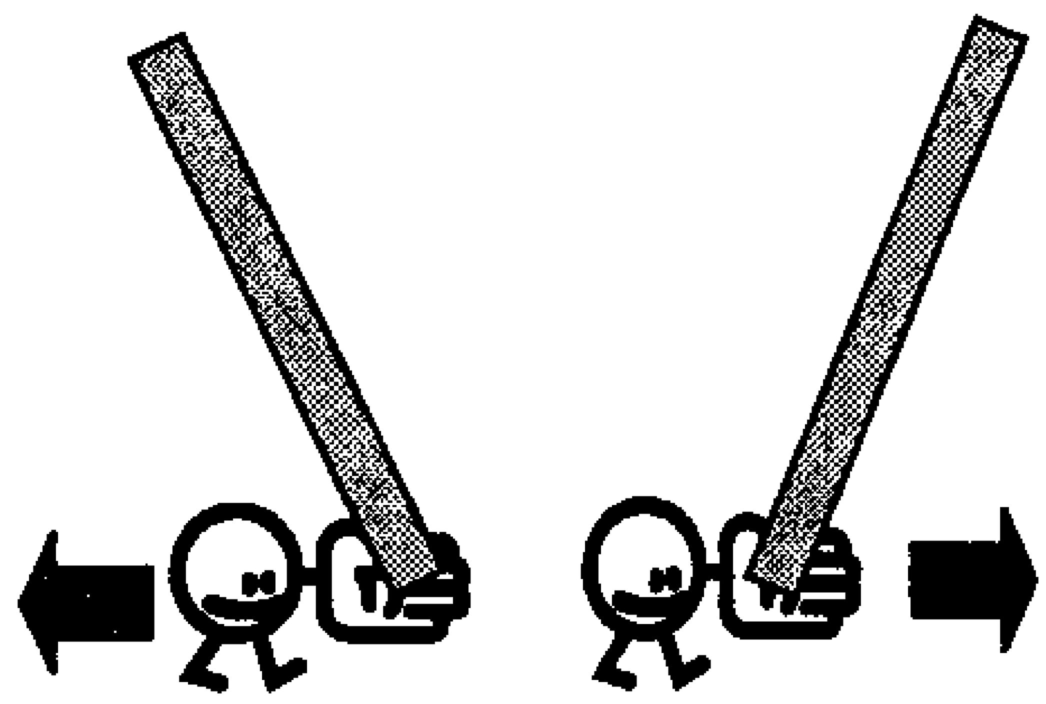


Figure 2 Fundamentals of how the inverted pendulum is stabilized

Purple arrows indicate the direction toward which the inverted pendulum should move when it is leaned as shown above.

certain force because either too fast motion or too slow motion fails the stabilization. When it starts to lean to the left, the pendulum's base moves left to make it upright again.

In definition, feedback control system monitors certain output of the system and manipulates its inputs to keep the output near to desired value. As seen in Figure 3, it is also called closed loop control system because the output affects the input which again produces the next output, repeating the cycle. It has several advantages over an open-loop controller it can quickly respond to the possible disturbances or uncertainties and keep the pendulum stabilized constantly.

In the closed loop controller, the desired output is called the reference. Difference between the reference and the current output is the error. The objective of the feedback control system is making it zero by manipulating inputs of the system. The diagram for feedback control system is shown as below.

In the case of the inverted pendulum, x, the position of the inverted pendulum, and theta, an angle between the rod and the perpendicular line, are outputs. Desired behavior, or reference, is theta = 0; we want to keep it upright. Their values are measured by instruments and put into the controller to calculate an input. The input is the force applied to the base of the pendulum, through which we manipulate its movement. However, the disturbance inputs, such as mechanical frictions, also involves in the motion of the inverted pendulum (Process), along with the input from the controller. Then, x and theta change, and these outputs again are measured and put into controller, completing one revolution of a closed-loop control system.

Open Loop Control System

Closed-Loop (a.k.a FEEDBACK) Control System

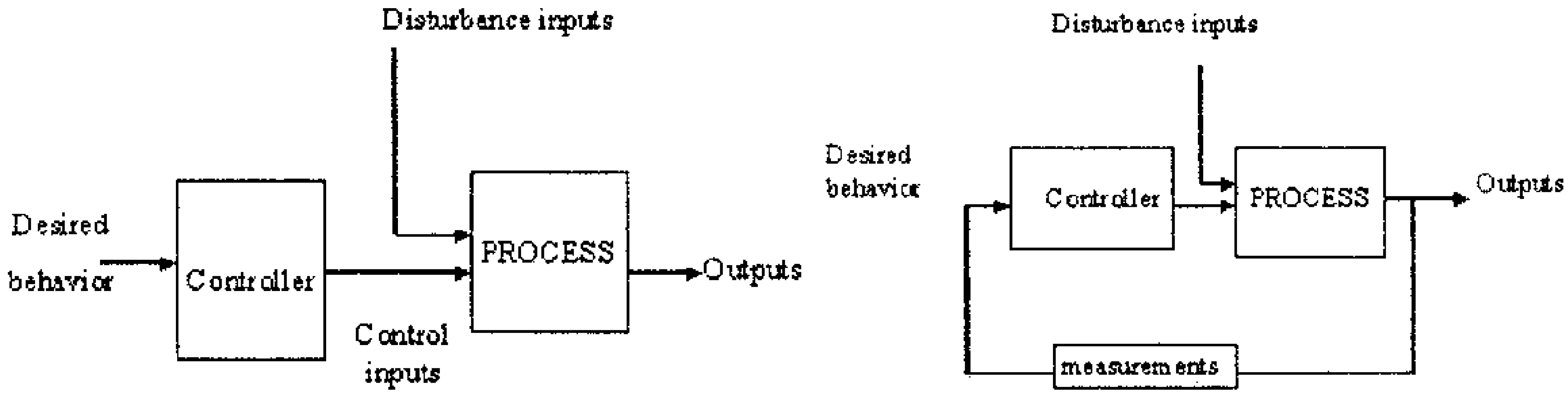


Figure 3 Comparison between Open Loop Control System and Closed Loop Control System[1]

Feedback control system differs from open loop system because it measures the output and feedback it to the controller. Output determines the value of the next output.

Among many kinds of controllers, a PID controller is a common feedback loop system. PID stands for Proportional, Integral, and Derivative. As the name implies, the input of the system is determined by three variations of the output's value; the error (Proportional), integral value of the error (Integral), derivative value of the error (Derivative). The equation for PID controller is shown as below.

$$\begin{aligned} Output &= P + I + D \\ &= K_P \cdot e + K_I \cdot \int e \cdot dt + K_D \frac{de}{dt} \qquad (19) \end{aligned}$$

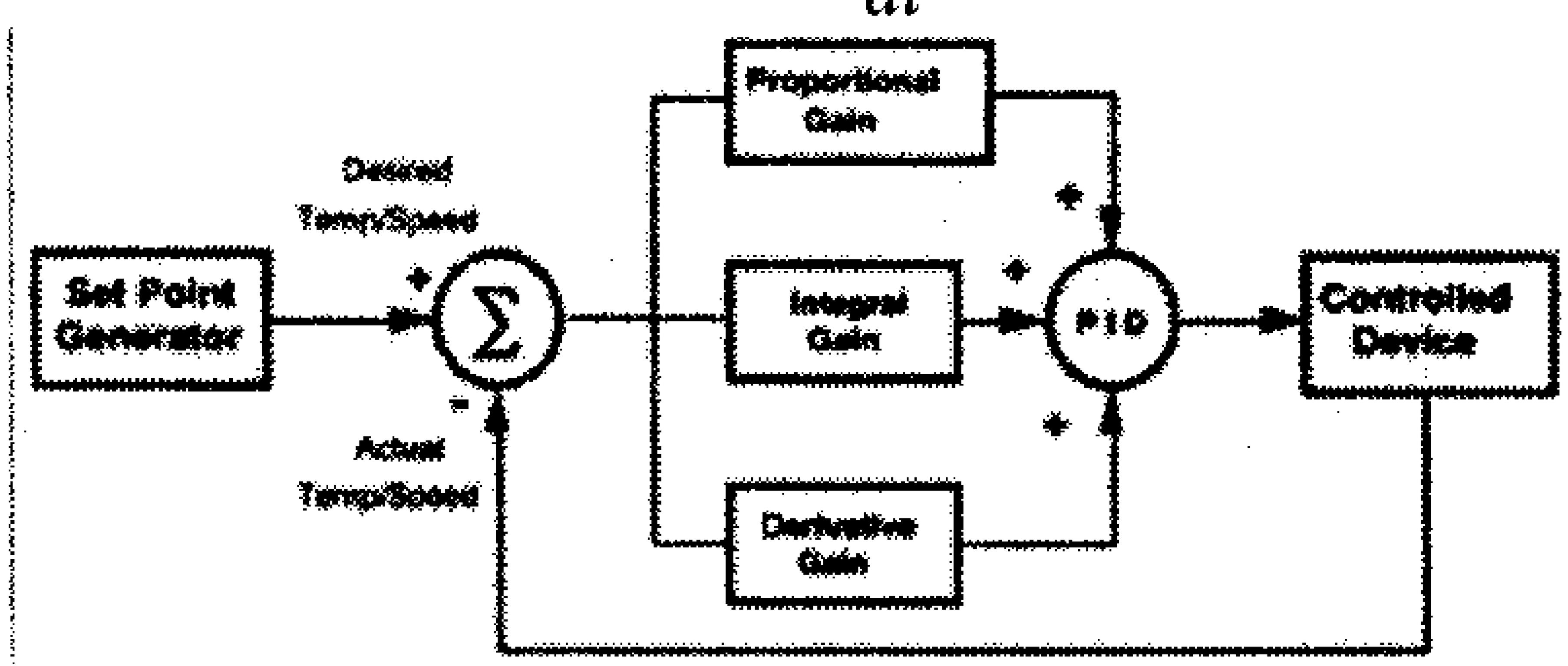


Figure 4 Diagram for PID Controller[2]

PID controller uses proportional gain, integral gain, and derivative gain of the error in order to determine the value of output which can make the error zero

This equation is needed to be analyzed in each component in physical points of view. In the inverted pendulum system, the error signifies how much the pendulum leans.

[1] http://www.ic-tech.com/Fuzzy%20Logic/
(Diagram 1: open 2: general 3. specified)

[2] http://www.brewerscience.com/products/cee/technical/ceepid/

a) Proportional component

The error is multiplied by K_p and added to the controlled output. For example, for a heater, a controller with a proportional band of 10 °C and a setpoint of 20 °C would have an output of 100% at 10 °C, 50% at 15 °C and 10% at 19 °C.[3]

In the pendulum system, when the rod leans more, the increased theta increases the sum of output. The result is that more force is applied to the pendulum's base. Note that when the error is zero, a proportional controller's output is zero.

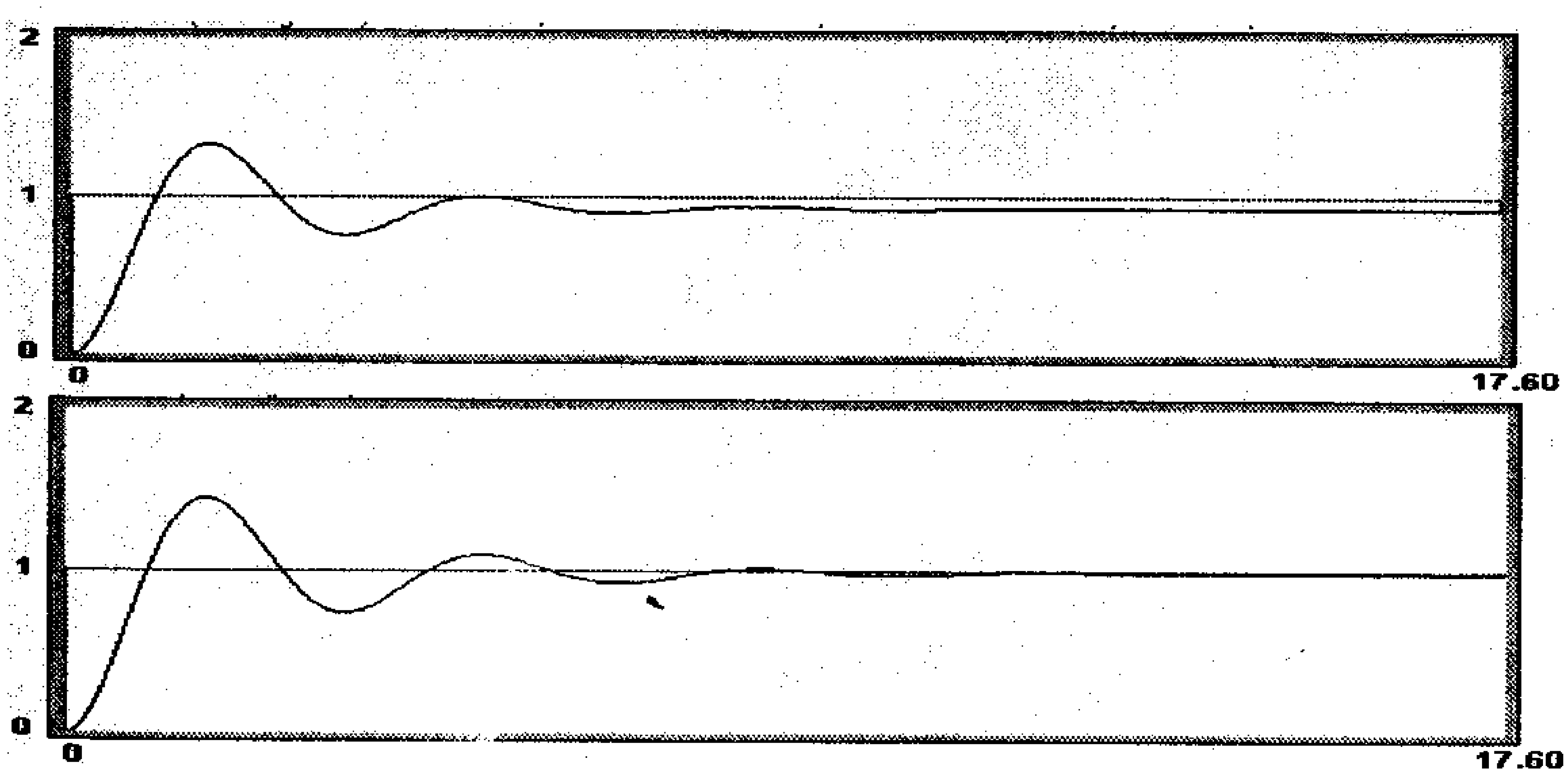
b) Integral component

Integral component signifies the average error during pendulum's movement. Integral movement extenuates too fast response from P and D components. From balance between I component and P/D component, the PID controller determines the patterns of stabilization.

c) Derivative component

Derivative of the error, or theta, is an angular velocity omega of the pendulum. It signifies how fast the pendulum is falling. The faster it falls, the more the force applied to it should be.

Each component contributes differently to the stabilization. Increase in proportional coefficient causes fast response, but it causes overshoots and steady state errors. Overshoots mean unnecessarily overt reaction. The highly responsive pendulum exerts too much force that the pendulum goes over the perpendicular line and fall down out of control. Also, steady state error is an error which is not removed by the controller. Bad controllers don't make error zero, and the value the error converges is steady state error. The integral component has a force which eliminates this steady state error. Although large value of integral gain may make oscillations even larger and make the system unstable, reducing the error improves the accuracy of the system. Lastly, derivative component also contributes to fast reaction and, more importantly, provides a dampening effect to eliminate oscillation and overshoots. In the inverted pendulum system, it provides an overall stability to it.



[3] Wikipedia : PID Controller

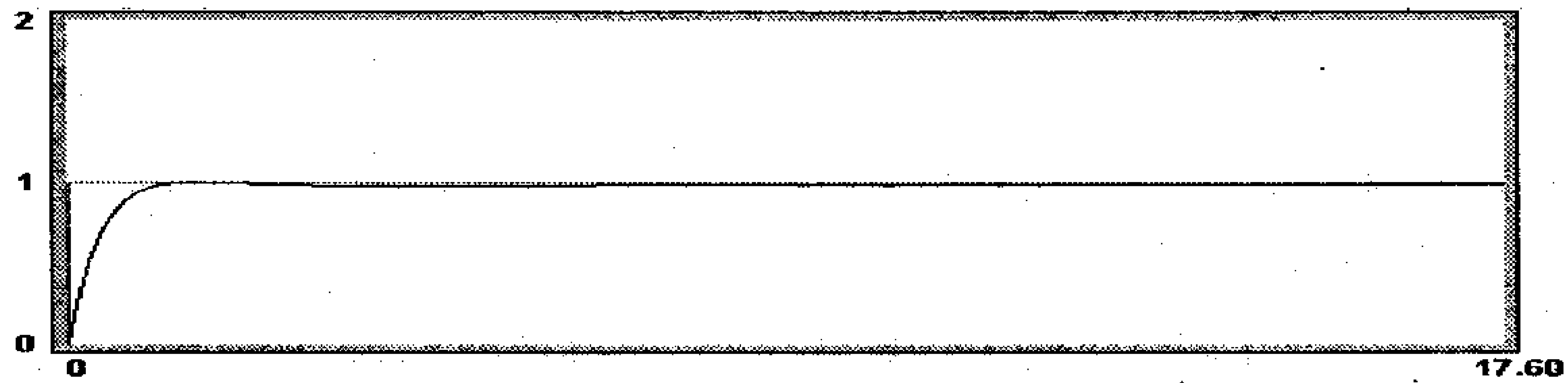


Figure 5 P Control, PI Control, PID Control (from top to bottom)[4]

P component is basic measure of the control system, I component eliminates steady state error, and D component extenuates oscillation.

Since each component has its good and bad points, one should combine the three in order to make the best stabilization of the pendulum. Finding ideal PID coefficients will be later explained in Discussion section.

iii) Difference between the two methods.

Both oscillation and feedback system can stabilize the inverted pendulum. However, their characteristics are different each other. In their mechanisms, simple oscillation is a kind of open-loop controller, whose input is fixed at any time. In contrast, feedback system is a complicated system whose input continuously changes depending on the pendulum's physical condition. As a result, it's unable to exactly predict the pendulum's motion.

The differences in physical complexity also determine the differences in the controlling device. Oscillating the pendulum needs a very simple machine such as a robot arm or a speaker (which makes very minute oscillations), but to realize feedback system needs an advanced electrical device which can perform differentiation and integration, and simulate mathematical modeling.

Nevertheless, feedback system is not an inefficient controller. Rather, the opposite is true. Complexity of the PID controller enables very flexible control compared to oscillation. In a strict sense, oscillating imperfectly stabilizes the pendulum because it doesn't make the rod stand upright but oscillate continuously. However, PID controller can vary the stabilizing movement by manipulating the values of P, I, D gains. In one case, one can make the stabilized pendulum oscillating precariously. In the other, one can make it exactly upright and not moving a bit.

Table 1 Comparison between oscillation and feed back control system

	Cost	Instruments	Complexity	Flexibility
Oscillation	**Less**	**Simple(speaker)**	**Can be solved with a clear-cut motion equation**	**Only one can stabilize or not.**
Feedback System	**More**	**Complex (a device with complicated arithmetic calculation)**	**Nonlinear Need to be familiar with control theory**	**Can vary the pattern of stabilization**

[4] Cuthbert Nyack Control (http://controlcan.homestead.com/files/acontrol/con2pid.htm)

C. Apparatus

Realizing the control system needs computer program which can simultaneously check the pendulum's conditions and calculate.

The inverted pendulum moves by the cart attached to it. The cart is fixed on a screw rail run by a motor. The cart and the pendulum do not move itself but the rail does. The motor is connected to the computer and it takes in charge of all the calculations needed for a PID controlling. The computer receives the information of x and theta, calculates how much force is needed to make the pendulum upright, and gives the value for torque applied to the motor. The motor's speed is determined by at every one thousandth second. Using CEMTool, a Matlab based program for controlling, we constructed PID controller system, which measures x and theta and calculates the force applied to the cart at every 0.001s.

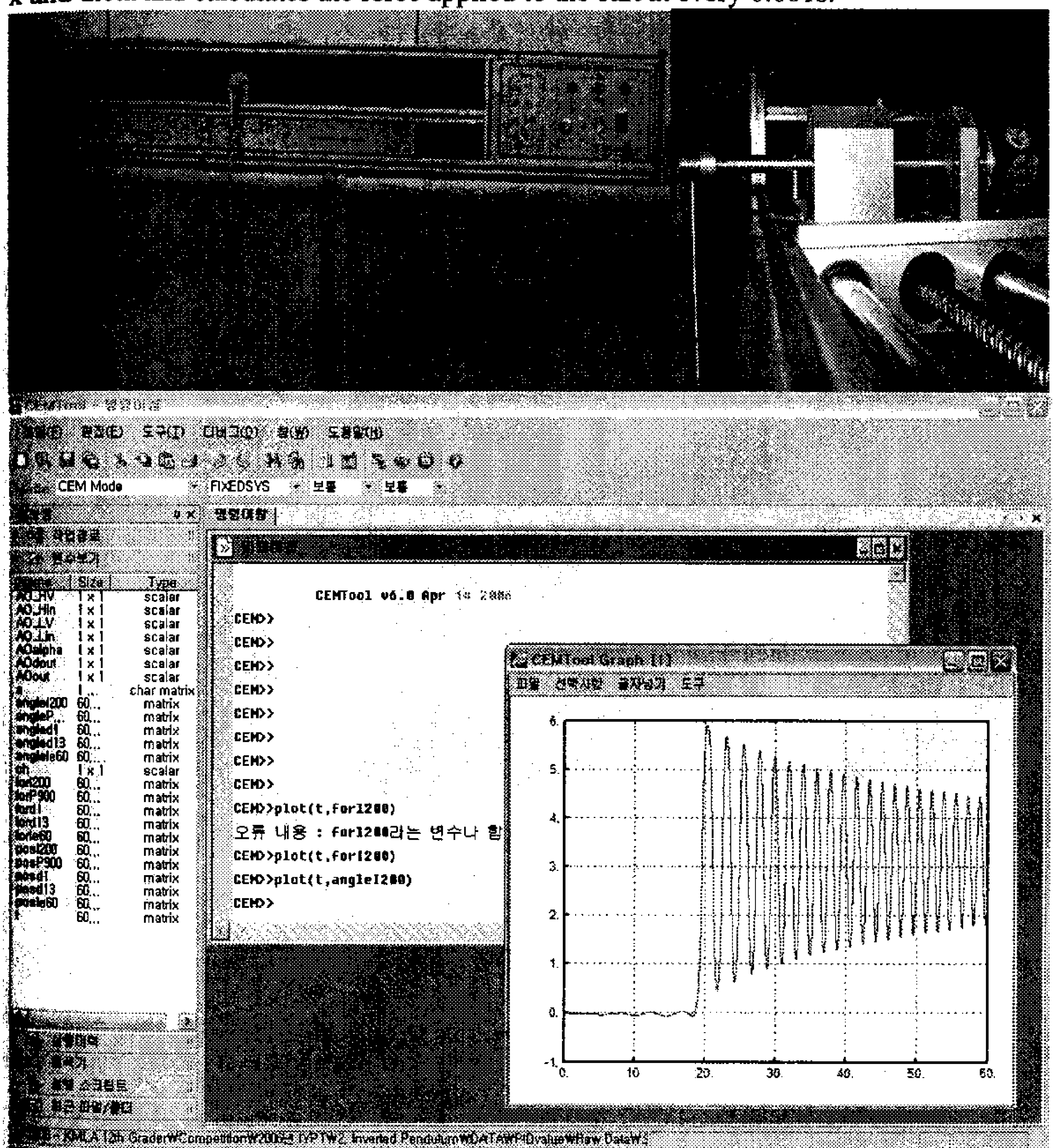


Figure 6 Apparatus of the Experiments (upper column) and Interface of CEMTool (lower column)

D. Experimental Method

Because of the limitation in the apparatus, we made experiments only on the case I and case iii.

i) horizontal oscillation

To realize the horizontal oscillation of the pendulum, we should know which function of the force is needed to make the pendulum oscillate precisely. However, the mechanics of the inverted pendulum system is complicated by the cart, the rail, and the motor which are not considered in the theoretical model. The precise physical model which takes those three into account is shown in Equation 20 & 21.

$$F = (m+M)\ddot{x} + b\dot{x} + ml\ddot{\theta}\cos\theta - ml\dot{\theta}^2\sin\theta \tag{20}$$

$$-\ddot{x}\cos\theta - g\sin\theta = 2l\ddot{\theta} \tag{21}$$

Basically, the real physical model has nonlinear components, so it's unable to get a solution for force F to make $x_0 = A\cos\omega t,\ y_0 = 0$

As a result, we needed to adopt PID controller in order to realize the horizontal oscillation. By handling PID gains, we made the pendulum stabilize with oscillation, and then began experiments. Although the process of setting the gains' values is arbitrary because of the system nonlinearity, we were able to vary the stabilization patterns using a conventional method for setting PID gains without mathematical calculation.[5]

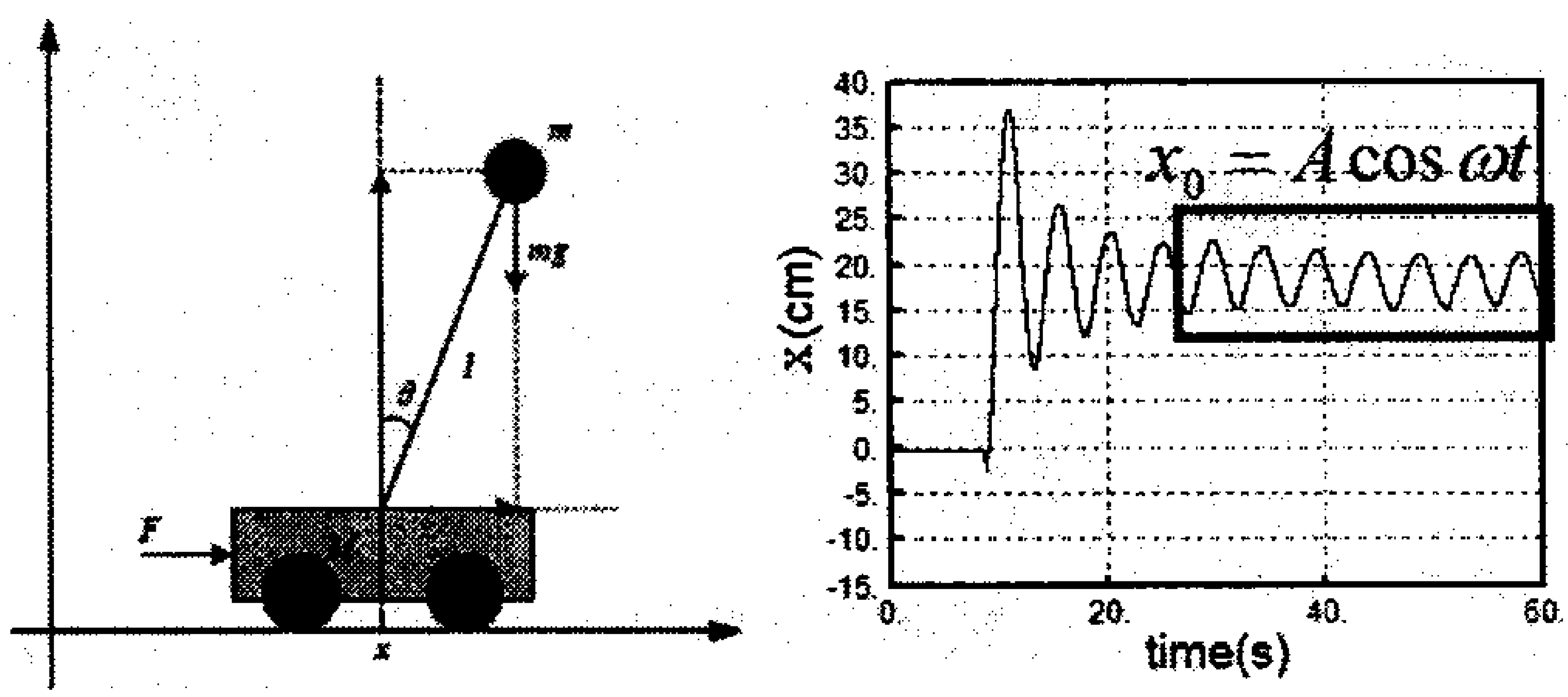


Figure 7 Diagram of cart-pendulum system (Left) and Oscillation interval of the controlled inverted pendulum (Right)

a) Experiment 1 : Proving the theory's validity

In the theoretical background part, we proved that in the stabilizing pendulum the value of theta is determined by x. From the graph of x, we drew the graph of theta calculated from Equation (18). Then, it is compared with the experimental value.

---

[5] Engineers in the control theory use more complicated method to determine PID gains, but we didn't use it because this paper focuses on the motion of the inverted pendulum, not the PID controller itself.

b) Experiment 2 : how mass and length affect stabilization

Among many variables which affect the stabilization of the inverted pendulum, its mass and length greatly determine major part of its motion, like those of the free-falling pendulum.

The pendulum consists of the heavy weight at the end and the rod. We prepared 4 different finds of weights (24.4g, 35.9g, 73.3g, 108.5g) and observed the difference in stabilizing pattern. Then again, with 4 different kinds of rods (30cm, 40cm, 60cm, 70cm) the experiments were repeated.

ii) Experiment 3 : PID Control

As mentioned before, with PID controller the inverted pendulum can have various stabilized patterns. It is also possible to stabilize faster and more accurately than simple oscillation. In this experiment, we searched for the condition of PID coefficients which can achieve the perfect stabilization of the inverted pendulum.

First, we changed the values of PID coefficients one by one and checked their effects each. In this case, we decided the standard of perfect stabilization as followings: first, less time to reach stabilization (less than 5 seconds) and, second, almost no oscillation after stabilization (angular amplitude less than 1cm). First, through simulation using CEMTool, we first grasped the range of PID coefficients which stabilizes the pendulum. Then, minute adjustment was based on actual demonstrations.

E.Result

i) Experiment 1 : validity of the theory

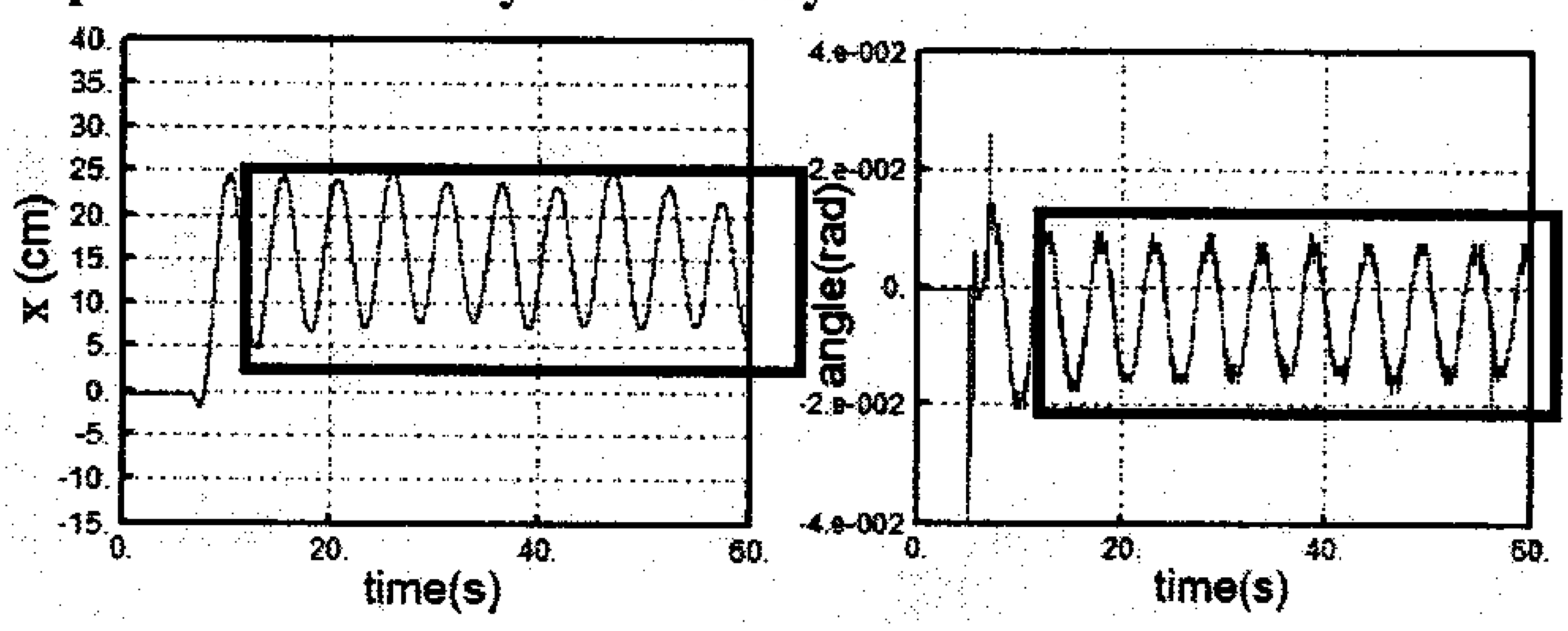


Figure 8 Graph of x and theta of the inverted pendulum stabilized by PID Controller

Black box shows the oscillating interval.

It was previously mentioned that when the inverted pendulum is imposed an oscillation $x_0 = A\cos\omega t,\ y_0 = 0$, theta becomes $\theta = -\dfrac{D\cos\omega t}{\omega^2 + \omega_0^2}$. Figure 8 is graphs for x_0 and theta measure from Experiment 1, and we can see that when x starts to follow a sinusoidal function, theta also resembles a sinusoidal function, implying the validity of the theoretical model. To prove that the the function of theta is exactly the same with that calculated by theoretical model, we should check the amplitude and angular frequency of the cosine fuction. Table 2, shown below, is the result for the comparison of angular frequency at various lengths and masses. Notice that length of the pendulum, shown in the table, is equal to L, not l, which is defined as the distance between the pendulum's pivot and the pendulum's center of mass. The relationship between L and l is later calculated in Appendix 1. m is the mass of the weight attached at the end of the pendulum.

Table 2 Comparison of Experimental Value and Theoretical Value of Angular Frequency

L (cm)	30	40	60	70
m (g)	108.5	108.5	108.5	108.5
$\omega_{\exp}$	1.213 ±0.081	1.206 ±0.040	1.230 ±0.061	1.206 ±0.053
ω_{theory}	1.199	1.220	1.199	1.215
Error	0.014	0.014	0.031	0.009
L (cm)	40	40	40	40
M (g)	24.4	35.9	73.3	108.5
$\omega_{\exp}$	1.188 ±0.060	1.213 ±0.051	1.236 ±0.063	1.239 ±0.062
ω_{theory}	1.198	1.207	1.218	1.229
Error	0.01	0.006	0.018	0.01

Table 3 Comparison of Experimental Value and Theoretical Value of Angular Amplitude

L (cm)	30	40	60	70
m (g)	108.5	108.5	108.5	108.5
$\theta^0_{\exp}$	11.257 ±0.909	7.068 ±0.785	5.498 ±0.785	5.149 ±1.277
θ^0_{theory}	11.596	7.221	6.049	4.633
Error	0.339	0.153	0.551	0.516
L (cm)	40	40	40	40
m (g)	24.4	35.9	73.3	108.5
$\theta^0_{\exp}$	19.350 ±0.785	17.618 ±0.453	12.043 ±0.906	6.977 ±0.453
θ^0_{theory}	19.984	18.070	11.865	7.427
Error	0.634	0.402	0.178	0.450

The error between expected amplitude and actual amplitude and that between expected frequency and actual frequency are little. From the result, we can see that the theoretical model fits well with the actual experiment. Finally, when the pendulum's pivot oscillates in $x_0 = A\cos\omega t,\ \ y_0 = 0$, it can be stabilized with theta $\theta = -\dfrac{D\cos\omega t}{\omega^2 + \omega_0^{\ 2}}$.

ii) Experiment 2 : how mass and length affect stabilization

So far, we saw that the theoretical model for the inverted pendulum is correct by comparing theoretical estimates and experimental values. Next, how the pendulum's physical characteristics, such as length and mass, affect the pattern of stabilization is studied during Experiment 2.

Figure 9 shows the result. For every length of the rod, a weight with 108.5g was used. As longer rod is used, the amplitude of the oscillation decreases while frequency keeps the same value.

Also, we varied the mass of the weight on the tip of the pendulum. For every mass of the rod, a 40-centimeter-long rod is used. As a weight with heavier mass is used, the amplitude of the oscillation decreases while frequency keeps the same value. One with a heavier mass tends to be stable.

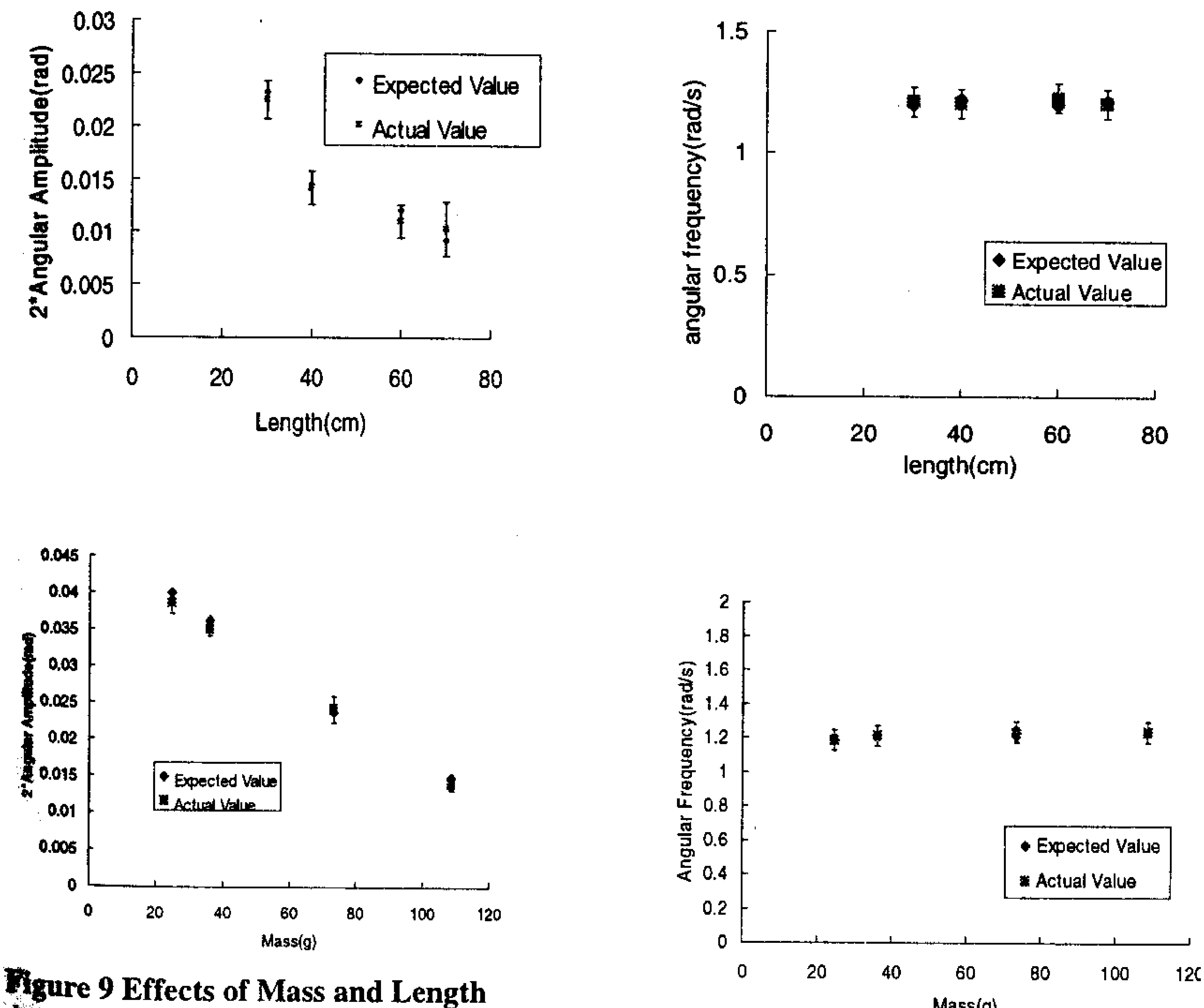


Figure 9 Effects of Mass and Length
As length increases, the amplitude of the cosine function decreases while the frequency remains the same. In other words, the long pendulum stabilizes more readily.

iii) Experiment 3 : PID Controller

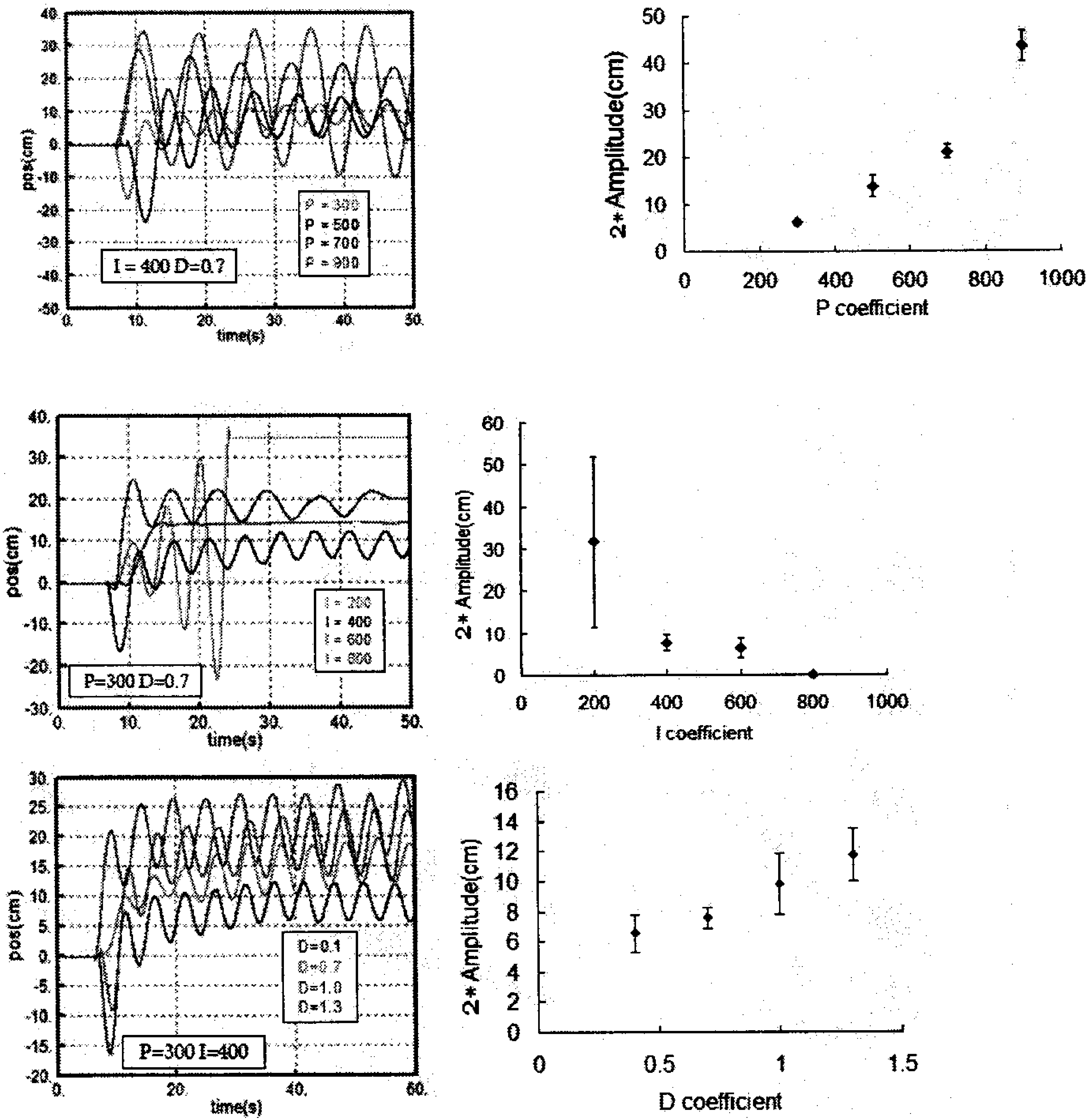


Figure 10 Effects of P, I, D Coefficients

Figure 10 shows how the pendulum's movement changes with P, I, D coefficients. Although the only obvious characteristic shown in the graph is amplitude, it still represents the coefficients' roles in stabilization. As P increases, the amplitude increases, which is very characteristic of P component. As I increases, the amplitude recognizably decreases, and especially when I = 800, the pendulum is stabilized with almost no oscillation. This case (P= 300, I = 800, D=0.7) fits to the conditions of perfect stabilization. As D increases, the amplitude increases. Difference between P and D is that the amplitude changes a lot more by D than by P. The next chapter will analyze the result from the two experiments.

F. Discussion

i) Experiment 2 : length

$$
\begin{aligned}
A_{ang} &= \frac{D}{\omega^2 + \omega_0{}^2} \\
&= ml\omega^2 A/(I\omega^2 + mgl) \\
&= m(\frac{2M+m}{2M+2m})L\omega^2 A/((M+\frac{1}{3}m)L^2\omega^2 + mg(\frac{2M+m}{2M+2m})L) \\
&= \frac{\alpha L}{\beta L^2 + \gamma L} = \frac{\alpha}{\beta L + \gamma} \qquad (22)
\end{aligned}
$$

where L is the length of the pendulum, l the distance between its pivot and its center of mass, m mass of the rod, M mass of the weight.

Substituting m, M, l, ω[6] for D and ω_0 clearly explains more stable motion in the longer pendulum. As seen in the last Equation (22), L is in the denominator so that increase in L eventually results in decrease in angular amplitude

In viewpoint from energy and torque, more length means more moment of rotation. Thus, the rotational energy from the basis affects less to the end of the long pendulum than that of the short pendulum. It results in less overshoots, making the pendulum more stable.

Constancy of the angular frequency is because angular frequency of theta is the same with frequency of x, defined by ω. Thus, length does not affect to the frequency. Note that the free falling pendulum's frequency is affected by length.

ii) Experiment 2 : mass

$$
A_{ang} = m(\frac{2M+m}{2M+2m})L\omega^2 A/((M+\frac{1}{3}m)L^2\omega^2 + mg(\frac{2M+m}{2M+2m})L) \qquad (23)
$$

From equation (23), increase in M causes angular amplitude decreases as denominator increases faster than numerator.[7] Thus, the graph in Figure 9 supports the theoretical expectation.

iii) Experiment 3 : PID Controller

The result from Experiment 3 basically supports the characteristics of P, I, D gains. Increase in P and D coefficients makes the response faster at cost of stability. As a result, the angular amplitude increases as the coefficients become larger. On the other hand, I coefficient improves the stability of the inverted pendulum system and reduces its angular amplitude.

We also saw the one case of perfect stabilization, in which the pendulum is stabilized with almost no oscillation. To find such cases of perfect stabilization, there are various methods, such as Ziegler-Nichols auto-tuning method and Relay auto-tuning method. However, in most cases, they are based not on physical modeling but mathematical control

[6] Detailed procedure of substitution is shown in the appendix 2.

[7] Detailed explanation for this statement is shown in the appendix 3.

theories. Although there are model-based methods such as root-locus method and transient response method, they are hard to apply to the inverted pendulum system because of its complexity. As explaining about the methods need comprehensive knowledge of control theories, we just introduce the name of methods.

G. Conclusion

The paper covers the inverted pendulum and its various controlling methods, including PID controller. Theoretical model was established based on the basic mechanism of the ordinary pendulum, and proved true by a series of experiments. It also enables us the prediction for how physical characteristics of the pendulum, such as mass and length, affects its movement and stability. The experiments showed the valid relationship between the pendulum's physical traits and stabilization patterns, which also correspond to the theoretical expectation. Finally, PID controller provides the possibility that the pendulum can not only just be stabilized but also be stabilized with various patterns. Among them, we analyzed the perfect stabilization case and realized it. In this aspect, PID controller shows high applicability and flexibility in controlling the inverted pendulum.

For one who should choose between PID controller or simple oscillation, the choice depends on one's preferences. As those two methods have strengths and weaknesses, shown in Table 1, one should choose strong points as a tradeoff for other strengths. In conclusion, the question how to keep the pendulum upright has many answers, and the choice is open to the users.

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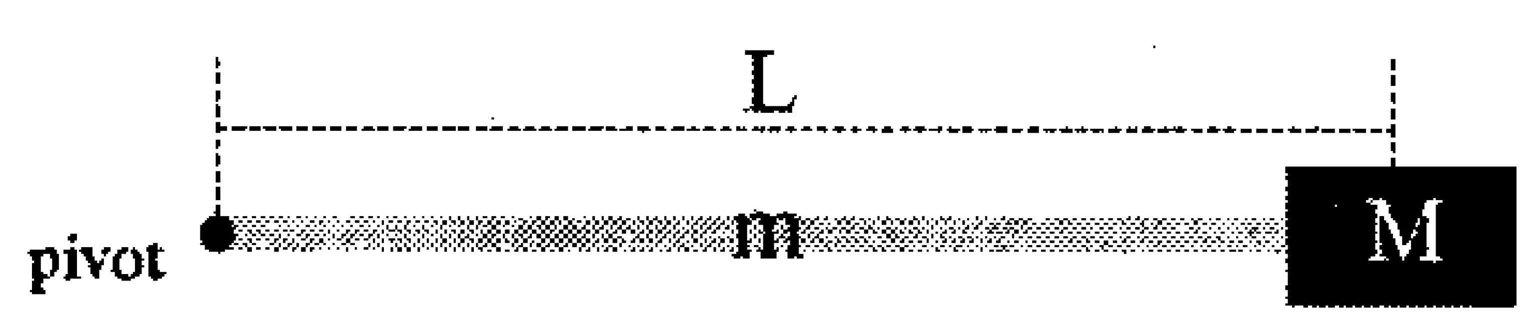
PID제어기 계수는 제어대상 시스템의 모델이 주어질 경우에 주파수영역 설계법, 근궤적법 등을 사용하여 반복과정을 통해 설계할 수 있다.

Appendices

i) Calculation of the Center of Mass and Rotational Inertia



a) **Center of Mass**

$$l = x_{CM} = \frac{\frac{1}{2}L \cdot m + L \cdot M}{m + M}$$
$$= \frac{2M + m}{2M + 2m} L$$

b) **Rotational Inertia**

$$I = I_{rod} + I_{mass}$$
$$= \frac{1}{3} mL^2 + ML^2$$
$$= (\frac{1}{3} m + M) L^2$$

ii) Detailed process of the equation for angular amplitude

$$A_{ang} = \frac{D}{\omega^2 + {\omega_0}^2}$$

where ${\omega_0}^2 = \frac{mgL}{I_p}$, $D = \frac{mL\omega^2 A}{I_p}$, $I = (\frac{1}{3} m + M) L^2$

When we all substitute w_o, D, and I, the equation becomes

$$A_{ang} = \frac{D}{\omega^2 + {\omega_0}^2}$$
$$= \frac{\frac{ml\omega^2 A}{I}}{\omega^2 + \frac{mgl}{I}}$$
$$= ml\omega^2 A / (I\omega^2 + mgl)$$
$$= m(\frac{2M + m}{2M + 2m}) L\omega^2 A / ((M + \frac{1}{3} m) L^2 \omega^2 + mg(\frac{2M + m}{2M + 2m}) L)$$
$$= \frac{\alpha L}{\beta L^2 + \gamma L} = \frac{\alpha}{\beta L + \gamma}$$

where $\alpha = m(\frac{2M + m}{2M + 2m})\omega^2 A$, $\beta = (M + \frac{1}{3} m)\omega^2$, $\gamma = mg(\frac{2M + m}{2M + 2m})$

iii) Denominator and numerator of the angular amplitude

The objective is to determine which side, denominator or numerator of the angular amplitude, increases faster as M increases. ω and A have their maximum value because the inverted pendulum cannot be stabilized over certain value of ω and A. Although the thresholds cannot be measure precisely, the experiments showed that approximate maximum value for ω and A is 1.5 and 0.3, respectively. Thus,

$$\omega^2 A < 1.5^2 * 0.3 < g \approx 9.8$$

Multiply the both sides by mL, then,

$$mL\omega^2 A < mgL$$

Thus, when x is variable, $mL\omega^2 Ax$ increases faster than $mgLx$. When $x = \frac{2M+m}{2M+2m}$, $m(\frac{2M+m}{2M+2m})L\omega^2 A$ increases faster than $mg(\frac{2M+m}{2M+2m})L$ as x increases. Also, $mg(\frac{2M+m}{2M+2m})L < (M+\frac{1}{3}m)L^2\omega^2 + mg(\frac{2M+m}{2M+2m})L$.

Thus, $m(\frac{2M+m}{2M+2m})L\omega^2 A$ increases faster than $(M+\frac{1}{3}m)L^2\omega^2 + mg(\frac{2M+m}{2M+2m})L$ as x increases

Since $x = \frac{2M+m}{2M+2m} = 1 - \frac{m}{2M+2m}$, x increases as M increases.

Thus, $m(\frac{2M+m}{2M+2m})L\omega^2 A$ increases faster than $(M+\frac{1}{3}m)L^2\omega^2 + mg(\frac{2M+m}{2M+2m})L$ as M increases

In other words, the denominator of A_{ang} increases faster than the numerator of A_{ang} as M increases.