7. Problem №10: Inverted pendulum

7.2. Solution of Ukraine

Problem №10: Inverted pendulum

Maxim Anisimov, UPhML KNU,
Ukrainian Physical and Mathematical Lyceum
of Kyiv National Taras Shevchenko University, Ukraine

The Problem:

It is possible to stabilize an inverted pendulum It is even possible to stabilize an inverted multiple pendulum /one pendulum the top of the other/. Demonstrate the stabilization and determine on which parameters this depends.

Introduction

Inverted pendulum is an interesting example of the non-linear oscillating system. The main idea of this device is an oscillating point of suspension. In this case, upper position of the pendulum is stable. The motion of the inverted pendulum is a classical example of the motion in the fast-oscillating field.

It was first theoretically investigated by P. L. Kapitza in 1951, so the physical model of the single inverted pendulum is well-known. Kapitza also demonstrated an experimental device to prove his theoretical results.

Problem statement.

To build a complete solution, both theoretical and experimental sides of the problem should be considered. Theory proves the possibility of stabilization, and experiment demonstrates it.

Physical model.

The main idea of such stabilization is to oscillate the point of suspension vertically with the frequency, lot more than the characteristic one. The motion equation looks as following (general form):

$$\ddot{x} + f(x) = F(x)\cos\Omega t\,, \qquad (1)$$

here $\Omega >> \frac{1}{T}$, where T is the specific period of motion in the system.

The solution X is found as the sum of slow and fast oscillating parts $x(t)$ and $\mu\chi(t)$. After some transformations the equation (1) is rewrote as:

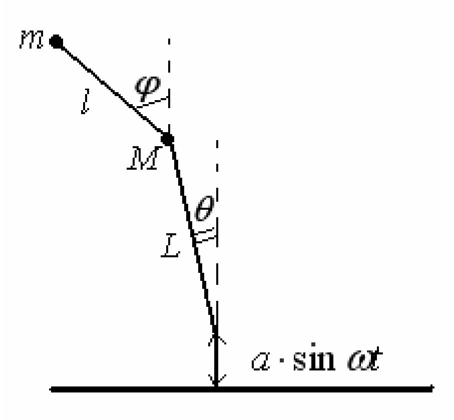
$$\ddot{X} + f(X) + \frac{1}{2}\frac{F(X)}{\Omega^2}\frac{\partial F}{\partial x}\bigg|_X = 0$$

We see that additional force appears; it is proportional to the oscillations' amplitude.

Now let's consider the double pendulum, consisting of two point loads on the weightless sticks:

First of all, we write down the coordinates of weights M and m:

$$x_M = L\sin\theta,$$
$$y_M = L\cos\theta + a\cdot\sin\omega t;$$
$$x_m = L\sin\theta + l\cdot\sin\varphi,$$
$$y_m = L\cos\theta + l\cdot\cos\varphi + a\cdot\sin\omega t;$$



Now let's evaluate the kinetic *(T)* and potential *(U)* energy of these weights:

$$T_M = \frac{M}{2}(\dot{x}_M^2 + \dot{y}_M^2),\ T_m = \frac{m}{2}(\dot{x}_m^2 + \dot{y}_m^2);$$
$$U_M = Mgy_M,\ U_m = mgy_m.$$

Then we write the Lagrange function (the difference *T-U*) for all the system:

$$\mathbf{L} = L^2\dot{\theta}^2\left(\frac{M+m}{2}\right) + a^2\omega^2\cos^2\omega t\left(\frac{M+m}{2}\right) - 2a\omega L\cos\omega t\dot{\theta}\sin\theta\left(\frac{M+m}{2}\right) +$$
$$+\frac{m}{2}\left(l^2\dot{\varphi}^2 + 2Ll\cos(\theta-\varphi)\dot{\theta}\,\dot{\varphi} - 2a\omega l\cos\omega t\sin\varphi\dot{\varphi}\right) - Lg\cos\theta(M+m) -$$
$$-ag\sin\omega t(M+m) - mgl\cos\varphi.$$

Now, using this expression, we can write down the following set of differential equations:

$$\frac{d}{dt}\left(\frac{\partial\mathbf{L}}{\partial\dot{\theta}}\right) = \frac{\partial\mathbf{L}}{\partial\theta},\quad \frac{d}{dt}\left(\frac{\partial\mathbf{L}}{\partial\dot{\varphi}}\right) = \frac{\partial\mathbf{L}}{\partial\varphi},$$

that describes the motion of double pendulum.

After derivation, simplifying and some transformations we get another system:

$$(M+m)\left[L\ddot{\theta} + a\omega^2\sin\omega t\sin\theta - g\sin\theta\right] + ml\left[\cos(\theta-\varphi)\ddot{\varphi} + \sin(\theta-\varphi)\dot{\varphi}^2\right] = 0$$
$$l\ddot{\varphi} + (a\omega^2\sin\omega t - g)\sin\varphi - L\sin(\theta-\varphi)\dot{\theta}^2 + L\cos(\theta-\varphi)\ddot{\theta} = 0\,.$$

***Equilibrium condition*.**

To solve these equations, we use the same method as for Kapitza pendulum equation. We substitute angles θ and φ in the form: $\theta = \gamma + \alpha,\ \ \varphi = \delta + \beta$, where the first terms vary slowly relatively to the second terms. Averaging upon the period of the point of suspension oscillations $\frac{2\pi}{\omega}$, one can get the set of 4 equations:

$$(M+m)\left[L\ddot{\gamma} + a\omega^2\langle\sin\omega t\cdot\alpha\rangle - g\gamma\right] + ml\ddot{\delta} = 0 \qquad \text{(2 a)}$$
$$l\ddot{\delta} + a\omega^2\langle\sin\omega t\cdot\beta\rangle - g\delta + L\ddot{\gamma} = 0 \qquad \text{(2 b)}$$

$$(M+m)\left[L\ddot{\alpha}+a\omega^2 \sin\omega t\cdot\gamma-g\alpha\right]+ml\ddot{\beta}=0 \qquad \text{(2 c)}$$

$$l\ddot{\beta}+a\omega^2 \sin\omega t\cdot\delta-g\beta+L\ddot{\alpha}=0 \qquad \text{(2 d)}$$

In the two last equations one should consider $\alpha = A\sin\omega t,\ \beta = B\sin\omega t$, and other variables are constant. Then, finding A and B from (2 c) and (2 d), one can get solutions in the form:

$A = f_1\gamma + f_2\delta$,

$B = f_3\gamma + f_4\delta$.

Then (2 a) and (2 b) look as follows:

$$L\ddot{\gamma}+\frac{a\omega^2}{2}(f_1\gamma+f_2\delta)-g\gamma+\mu l\ddot{\delta}=0,$$

$$l\ddot{\delta}+\frac{a\omega^2}{2}(f_3\gamma+f_4\delta)-g\delta+L\ddot{\gamma}=0,\ \mu=\frac{m}{M+m}.$$

We find general solution in the exponential form, $\gamma \sim \delta \sim e^{\lambda t}$.

After some transformations we get the characteristic equation for λ:

$$\gamma\delta\left(\lambda^2 L+\frac{a\omega^2}{2}f_1-g\right)\left(-l\lambda^2-\frac{a\omega^2}{2}f_4+g\right)=\delta\gamma\left(-\frac{a\omega^2}{2}f_2-\mu l\lambda^2\right)\left(L\lambda^2+\frac{a\omega^2}{2}f_3\right)$$

It is biquadrate, and possible roots are $\pm\lambda_1, \pm\lambda_2$. For the stable equilibrium they should be purely imaginary, so, $\left(\lambda^2\right)_{1,2}<0$. Writing the equation as polynomial of λ, one can get:

$$\lambda^4(Ll-\mu lL)+\lambda^2\left(\frac{La\omega^2 f4}{2}-Lg+\frac{a\omega^2 f1\, l}{2}-gl-\frac{a\omega^2 f2\, L}{2}-\frac{\mu l a\omega^2 f3}{2}\right)$$
$$+\frac{a^2\omega^4 f1\, f4}{4}-\frac{a\omega^2 f1\, g}{2}-\frac{ga\omega^2 f4}{2}+g^2-\frac{a^2\omega^4 f2\, f3}{4}=0$$

or

$$A\lambda^4+B\lambda^2+C=0.$$

We designate coefficients near λ by A,B,C, where A>0. According to the Wiet theorem, to find the equilibrium, the following conditions should be fulfilled:

$B>0$

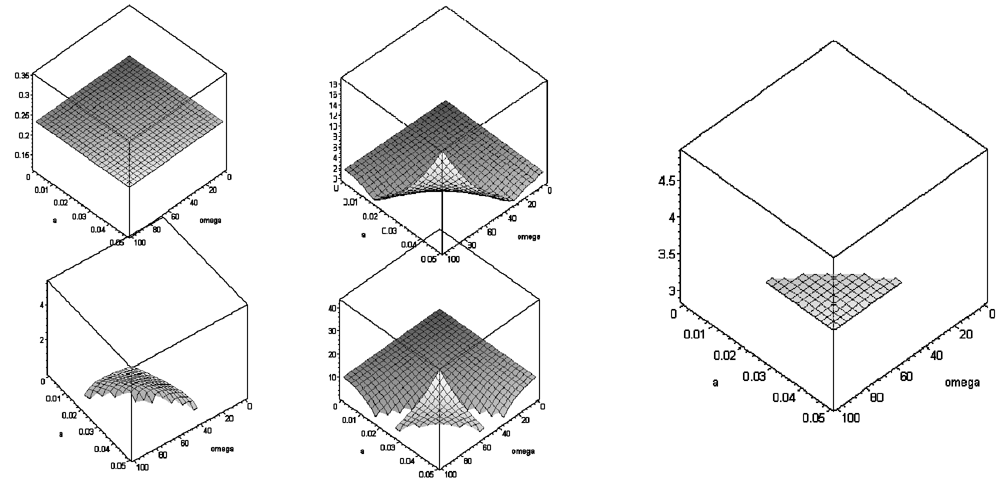
$C>0$

$D=B^2-4AC>0$

It's difficult to solve these inequations manually, but with the help of computer it's possible to get numerical dependences of *A,B,C,D* on the frequency and amplitude.

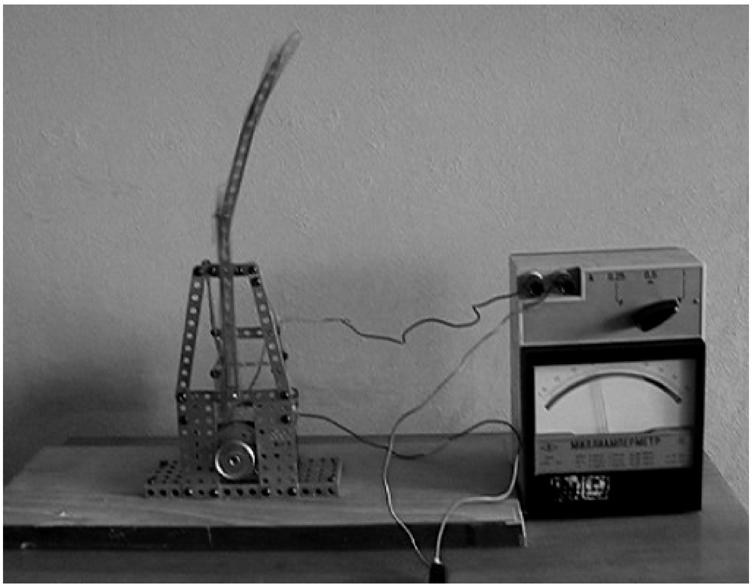
Graphs showing dependence upon amplitude and frequency,

built with Maple 8 (Constants: $M=0.1;\ m=0.01;\ L=0.3;\ l=0.2$).



***Experiment*.**

To demonstrate the real stabilization, our team constructed an experimental device. Its photo is shown below:



***Conclusions*.**

To solve the problem theoretically, known physical model of a single inverted pendulum was considered. It was used as a basic one to build the theory of double pendulum. Resulting solution determines the possibility of stabilization, depending on initial conditions, which are inputted numerically.

The presented problem solution demonstrates an example of the physical modelling of non-linear oscillating systems.