

PROBLEM No.12 ROLLING MAGNET

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- Abstract -

The study analyzed the motion of a magnet which rolled down an inclined copper plate based on the motion equation and the model of eddy currents which were able to explain the motions on the x-axis and the y-axis respectively considering the compositive effects of gravitational force, frictional force, the electromagnetic forces exerted among magnetic dipoles(between a magnet and the magnetic field generated by eddy current, and between a magnet and earth's magnetic field).

Specifically, it examined that the restitution force, F exerted on the y-axis was proportional to y/ω^3 , and analyzed the oscillation of a magnet quantitatively. The study analyzed the motion of a rolling magnet on the x-axis and the y-axis according to the change of an inclined angle(θ), and calculated the constants in the motion equation. Also, it explained and analyzed quantitatively the shift of the center line of oscillation and that of the phase according to the change of an angle between the x-axis(the direction of magnet's movement) and the North Magnetic Pole(ϕ). Lastly, it examined that the relationship between the period(T) and the width of a copper plate(ω) is $T^2 \propto \omega^3$

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I . Introduction

A neodymium(Nd) magnet is much stronger than the ferrite magnet that people commonly use, so a Nd magnet is very sensitive to the external magnetic field. Because of this characteristic of a Nd magnet, it arranges easily to the direction of the earth's magnetic field, and induces a very strong eddy current when it moves above a metal plate.

The Nd magnet, which has these characteristics, is expected to move in special paths due to the combined effect of electromagnetic components like eddy currents and the earth's magnetic field and that of mechanic components like gravitational force and frictional force. Also, it is expected to analyze the concept of eddy currents quantitatively which has been familiar as a qualitative concept by approximately solving the motion equation of a magnet.

The research is to examine how the earth's magnetic field and magnetic field induced by eddy currents affect to the motion of a rolling Nd magnet, and to know what kinds of paths a magnet follows.

These are the purposes of my research:

1. Examine various motions of a magnet which rolls down an inclined metallic plate.
2. Measure the magnet's displacement per time, and analyze the oscillating motion of a magnet examining the change of its amplitude, period and wavelength.
3. Understand the effect of an inclined angle of a metallic plate(θ), the width of a plate(ω), an angle between the x-axis(the direction of magnet's movement) and the North Magnetic Pole(ϕ) and the initial position of a magnet on a plate to the motion of a magnet.
4. Set the motion equation of a rolling magnet and the model of eddy currents which are able to explain the motion of a rolling magnet, and analyze the motion using them.
5. Understand how the earth's magnetic field and the magnetic field induced by eddy currents affect to the motion of a rolling magnet.

II. Theoretical Background

2-1. Vector potential of the dipole term

For convenience, let us give the symbol \mathbf{D} to the integral in the dipole term :

$$\mathbf{D} = \int_V \mathbf{J}(\mathbf{r}')(\hat{\mathbf{r}} \cdot \mathbf{r}') d\tau' \quad (1)$$

The process of writing this as a product in which the field point is somehow separated out is somewhat involved and it is easier to deal with a scalar quantity. If we let \mathbf{C} be any arbitrary constant vector, then we can form the scalar product $\mathbf{C} \cdot \mathbf{D}$. After doing this, we divide the integrand into two equal parts, add and subtract the quantity $\frac{1}{2}(\mathbf{J} \cdot \hat{\mathbf{r}})(\mathbf{C} \cdot \mathbf{r}')$ under the integral, and we find that we can write the result as the sum of two integrals:

$$\mathbf{C} \cdot \mathbf{D} = \int_V (\mathbf{C} \cdot \mathbf{J})(\hat{\mathbf{r}} \cdot \mathbf{r}') d\tau' = (\mathbf{C} \cdot \mathbf{D})_+ + (\mathbf{C} \cdot \mathbf{D})_- \quad (2)$$

where

$$(\mathbf{C} \cdot \mathbf{D})_+ = \frac{1}{2} \int_V [(\mathbf{C} \cdot \mathbf{J})(\hat{\mathbf{r}} \cdot \mathbf{r}') + (\mathbf{J} \cdot \hat{\mathbf{r}})(\mathbf{C} \cdot \mathbf{r}')] d\tau' \quad (3)$$

$$(\mathbf{C} \cdot \mathbf{D})_- = \frac{1}{2} \int_V [(\mathbf{C} \cdot \mathbf{J})(\hat{\mathbf{r}} \cdot \mathbf{r}') - (\mathbf{J} \cdot \hat{\mathbf{r}})(\mathbf{C} \cdot \mathbf{r}')] d\tau' \quad (4)$$

and we consider there separately.

Now \mathbf{C} is a constant by definition, and $\hat{\mathbf{r}}$ is a constant with respect to source point derivatives, that is, with respect to ∇' . Therefore,

$$\nabla'(\mathbf{C} \cdot \mathbf{r}') = \mathbf{C} \quad \text{and} \quad \nabla'(\hat{\mathbf{r}} \cdot \mathbf{r}') = \hat{\mathbf{r}} \quad (5)$$

Using these last results, we find that the bracketed part of the integrand of (3) can be written as

$$\begin{aligned} \mathbf{J} \cdot [\mathbf{C}(\hat{\mathbf{r}} \cdot \mathbf{r}') + \hat{\mathbf{r}}(\mathbf{C} \cdot \mathbf{r}')] &= \mathbf{J} \cdot [(\hat{\mathbf{r}} \cdot \mathbf{r}')\nabla'(\mathbf{C} \cdot \mathbf{r}') + (\mathbf{C} \cdot \mathbf{r}')\nabla'(\hat{\mathbf{r}} \cdot \mathbf{r}')] \\ &= \mathbf{J} \cdot \nabla'[(\mathbf{C} \cdot \mathbf{r}')(\hat{\mathbf{r}} \cdot \mathbf{r}')] \end{aligned} \quad (6)$$

Now the last term has the form $\mathbf{J} \cdot \nabla' \mathfrak{J}$ where \mathfrak{J} is the scalar $(\mathbf{C} \cdot \mathbf{r}')(\hat{\mathbf{r}} \cdot \mathbf{r}')$; thus, we can write (6) further as

$$\mathbf{J} \cdot \nabla' \mathfrak{J} = \nabla' \cdot (\mathfrak{J} \mathbf{J}) - \mathfrak{J}(\nabla' \cdot \mathbf{J}) = \nabla' \cdot [(\mathbf{C} \cdot \mathbf{r}')(\hat{\mathbf{r}} \cdot \mathbf{r}') \mathbf{J}] \quad (7)$$

which is our final form for the integrand of (3), we finally get

$$(C \cdot \nabla)_+ = \frac{1}{2} \oint_{S'} (C \cdot r') (\hat{r} \cdot r') (J \cdot da') \quad (8)$$

where S' is the surface enclosing V' . However, V' encloses all of the currents so that $J = 0$ at all area elements da' of S' , and therefore

$$(C \cdot \nabla)_+ = 0 \quad (9)$$

The terms in the brackets in (4) can be written as

$$C \cdot [J(\hat{r} \cdot r') - r'(J \cdot \hat{r})] = C \cdot [r' \times (J \times r')] \quad (10)$$

If we put this into (4), and use (5), we find that (3) becomes

$$C \cdot \nabla = \frac{1}{2} \int_{V'} C \cdot [\hat{r} \times (J \times r')] d\tau' = C \cdot \left\{ \frac{1}{2} \int_{V'} [r' \times (J \times r')] d\tau' \right\} \quad (11)$$

Since C is a completely arbitrary vector, (11) can be always true only if D equals the quantity in braces. Furthermore, \hat{r} is constant as far as the integration over the primed variables is concerned and can be taken out of the integral. Doing this, we find that

$$D = \hat{r} \times \frac{1}{2} \int_{V'} J \times r' d\tau' = \left(\frac{1}{2} \int_{V'} r' \times J d\tau' \right) \times \hat{r} \quad (12)$$

Finally, if we combine (12), (11) and the general multipole expansion of the vector potential¹⁾, we find that the dipole term can be written as

$$A_D(r) = \frac{\mu_0}{4\pi r^2} \left[\frac{1}{2} \int_{V'} r' \times J(r') d\tau' \right] \times \hat{r} \quad (13)$$

which has the desired form of a product of a quantity referring only to the location of the field point and something depending only on the properties of the current distribution.

The quantity in brackets is given the symbol

$$m = \frac{1}{2} \int_{V'} r' \times J(r') d\tau' \quad (14)$$

1) $A(r) = \frac{\mu_0}{4\pi} \sum_{l=0}^{\infty} \frac{1}{r^{l+1}} \int_{V'} J(r') r'^l P_l(\cos \Theta') d\tau'$

and is called the magnetic dipole moment of the current distribution. This definition enables us to write the dipole term as

$$A_D(r) = \frac{\mu_0}{4\pi} \frac{\mathbf{m} \times \hat{\mathbf{r}}}{r^2} = \frac{\mu_0}{4\pi} \frac{\mathbf{m} \times \mathbf{r}}{r^3} \quad (15)$$

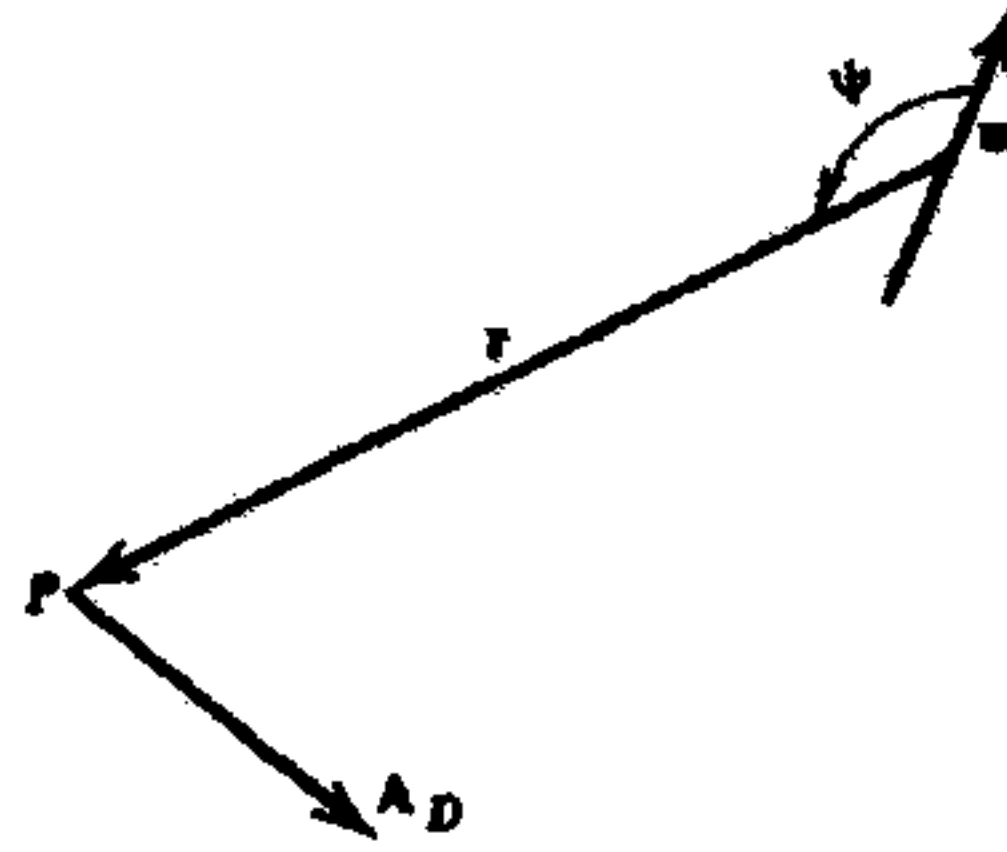


Figure 1. Relation between a magnetic dipole and the vector potential it produces

As shown in Figure 1, A_D is perpendicular to the plane formed by \mathbf{m} and \mathbf{r} and its magnitude will be $A_D = \mu_0 m \sin \Psi / 4\pi r^2$.

2-2. The magnetic dipole field

The expression A_D given by (15) will be the predominant term in the vector potential when the field point is sufficiently far away from the current distribution. In order to study its properties it is convenient, to assume that (15) holds everywhere in space. Then we can call it the magnetic dipole field and think of it as being produced by a frictional point dipole \mathbf{m} located at the origin. Later we will consider specific current distributions that can be thought of in this way, but for now we concentrate on finding the induction \mathbf{B} produced. We use spherical coordinates to locate the field point P and choose the z axis in the direction of \mathbf{m} ; this gives the situation shown in Figure 2-1. Then $\mathbf{m} = m\hat{\mathbf{z}}$ and (15) becomes

$$A_D(r) = \frac{\mu_0 m}{4\pi r^2} \hat{\mathbf{z}} \times \hat{\mathbf{r}} = \frac{\mu_0 m \sin \Theta}{4\pi r^2} \hat{\Phi} \quad (16)$$

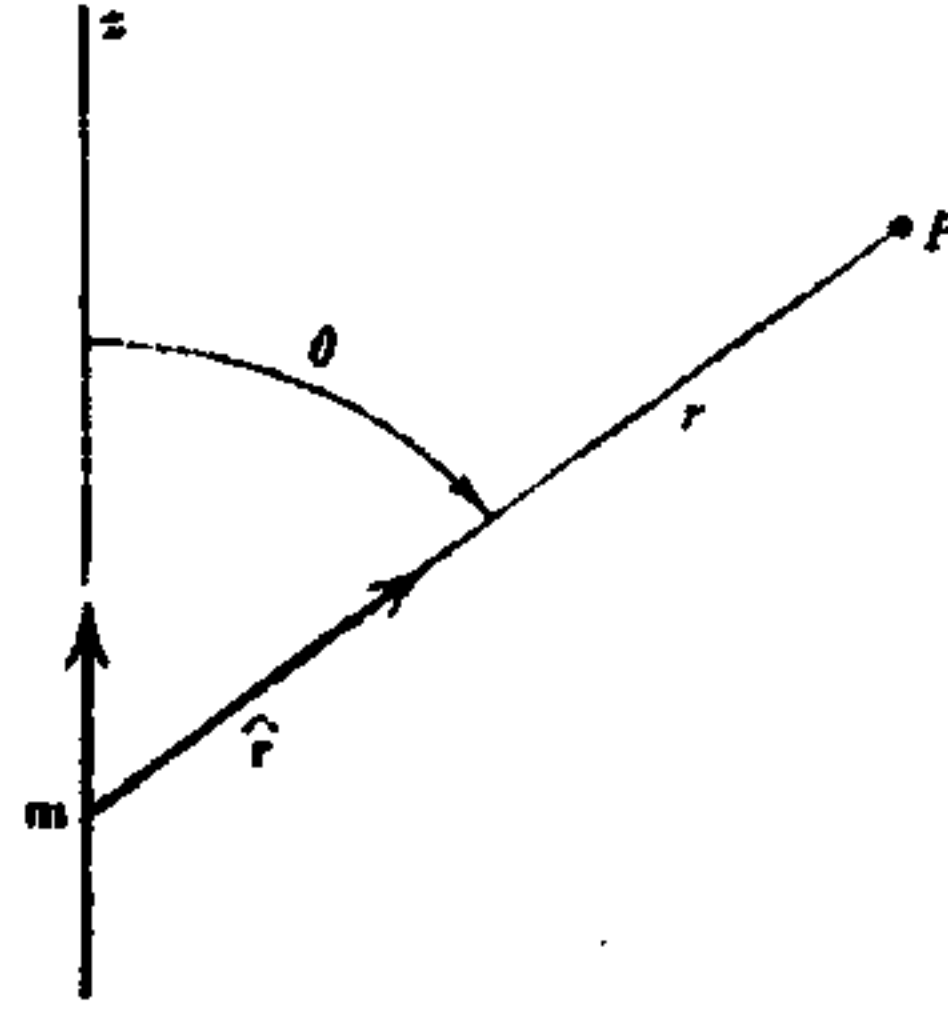


Figure 2. Calculation of the field produced by a magnetic dipole

so that the only non-zero component of A_D is $A_{D\phi}$. The curves of constant values of $A_{D\phi}$ are given by

$$r^2 = \left(\frac{\mu_0 m}{4\pi A_{D\phi}} \right) \sin \Theta = C_D \sin \Theta \quad (17)$$

where the constant C_D characterizing a given curve depends on the value of $A_{D\phi}$. These curves are between as the solid lines in Figure 3; however, the direction of $A_{D\phi}$ is into the page on the right half of the figure and out of the page for the left half.

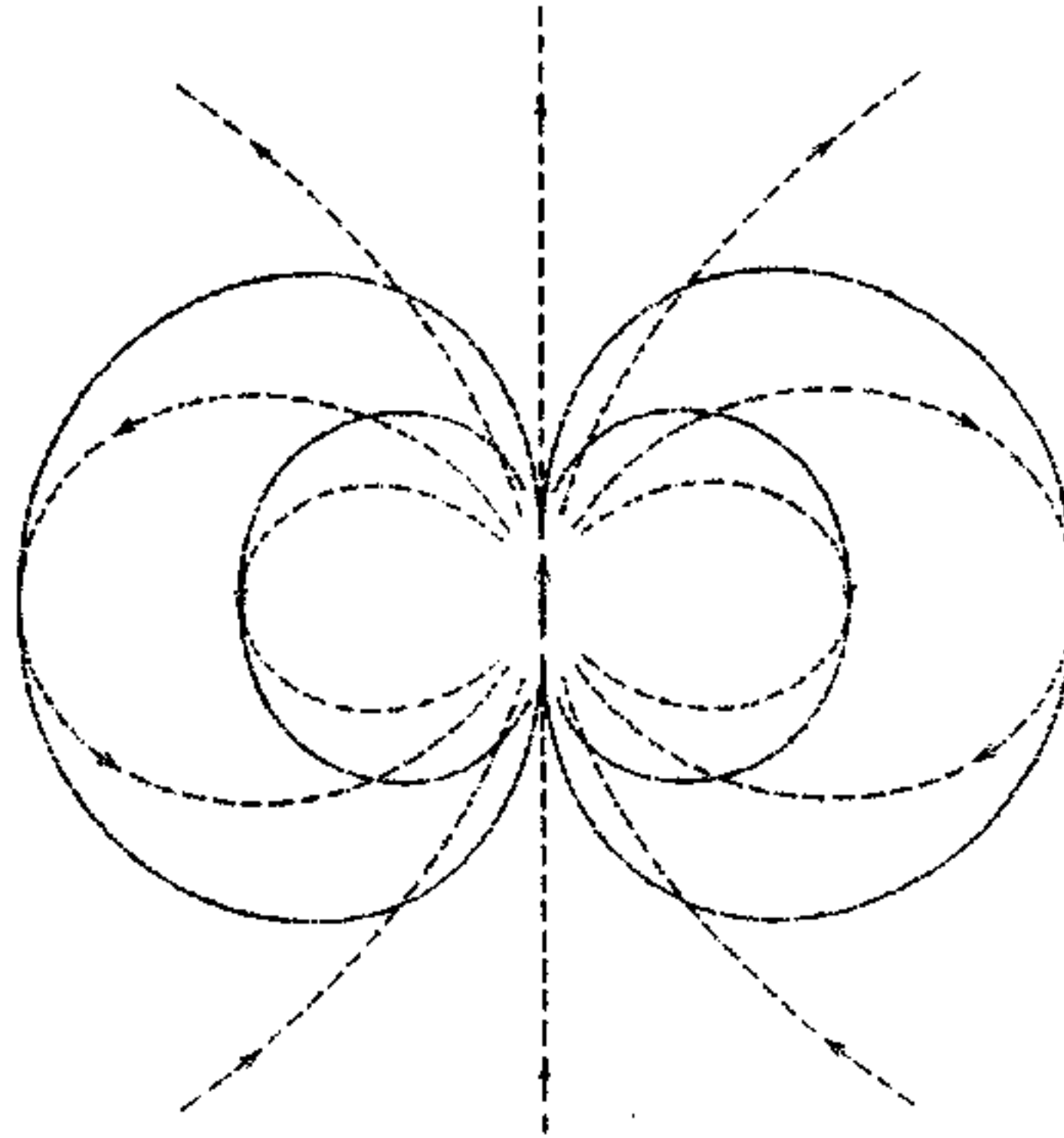


Figure 3. Field of a magnetic dipole. Lines of B are dashed. Solid lines are those of constant magnitude of the vector potential that is directed into the page for the right half and out of the page for the left half

The induction is found from $\mathbf{B} = \nabla \times \mathbf{A}_D$. Using (14), we find the components of \mathbf{B} to be

$$B_r = \frac{1}{r \sin \Theta} \frac{\partial}{\partial \Theta} (\sin \Theta A_{\Theta}) = \left(\frac{\mu_0 m}{4\pi} \right) \frac{2 \cos \Theta}{r^3} \quad (18)$$

$$B_{\Theta} = -\frac{1}{r} \frac{\partial}{\partial \Theta} (r A_{\Theta}) = \left(\frac{\mu_0 m}{4\pi} \right) \frac{\sin \Theta}{r^3} \quad (19)$$

while $B_{\phi} = 0$. Thus \mathbf{B} lies in the same plane as \mathbf{m} and the field point, while A_{Θ} is perpendicular to this plane

From this results, we can see that they are proportional to the magnitude of their respective dipole moments and they have the same dependence on angle and distance.

2-3. dipole energy

If we write it simply as U_D , then the energy of a dipole \mathbf{m} in an external field \mathbf{B}_0 is

$$U_D = -\mathbf{m} \cdot \mathbf{B}_0 = -mB_0 \cos \Psi \quad (20)$$

where Ψ is the angle between them as shown in Figure 4.

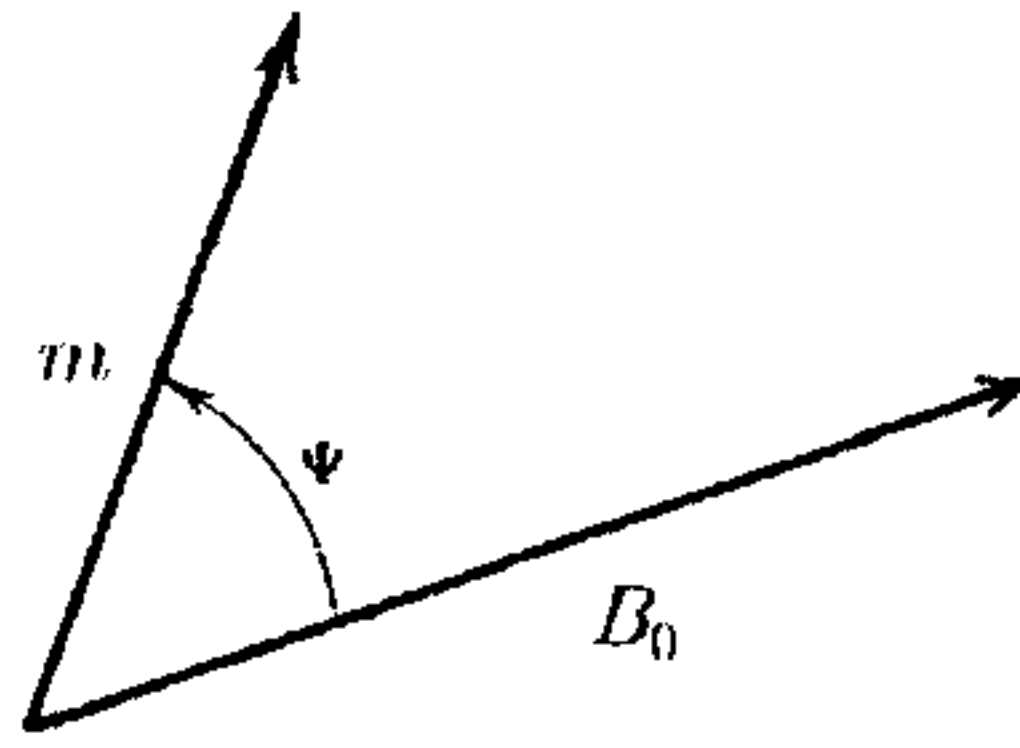


Figure 4. A dipole making an angle with an external magnetic field

Since the energy is a function of Ψ , that is, $U_D = U_D(\Psi)$, we know from mechanics that there will be a torque on \mathbf{m} , and the component of this torque in the direction of increasing Ψ , τ , will be given by

$$\tau = -\frac{\partial U_D}{\partial \Psi} = -mB_0 \sin \Psi = -|\mathbf{m} \times \mathbf{B}_0| \quad (21)$$

Since τ is negative, it means that the sense of the torque on \mathbf{m} is such as to rotate \mathbf{m} into the direction of \mathbf{B}_0 ; thus, if we use the definition of the direction of the cross product, we can see that the vector torque $\boldsymbol{\tau}$ is in the direction of $\mathbf{m} \times \mathbf{B}_0$ as shown in Figure 4. Combining with (21), the torque on \mathbf{m} is given correctly in magnitude and direction by

$$\boldsymbol{\tau} = \mathbf{m} \times \mathbf{B}_0 \quad (22)$$

III. The plan for experiments

To examine how the earth's magnetic field, the magnetic field induced by eddy currents which are generated by a rolling magnet, gravitational force and frictional force, affect to the motion of a disk shaped magnet($\phi 20 \times 5mm$, 12g) which rolls down an inclined metallic plate, experimental set-ups are set like Figure 5.

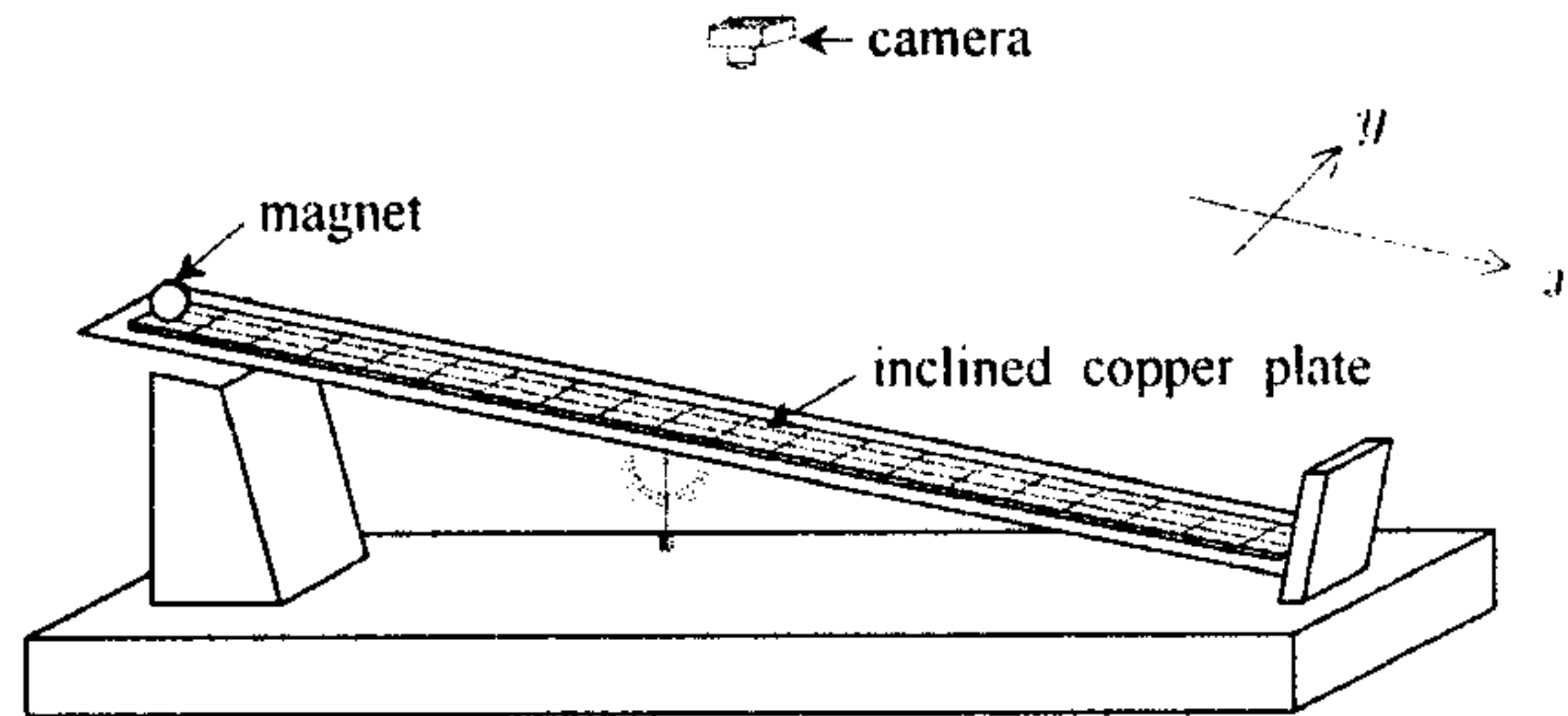


Figure 1. The diagram of experimental set-ups

3-1. Parameters of experiments

Parameters of experiments are an inclined angle of a metallic plate(θ), the width of a plate(ω), an angle between the x-axis(the direction of magnet's movement) and the North Magnetic Pole(ϕ), and the initial position of a magnet on a plate.

3-1-1. A metallic plate

This is a copper plate where a disk shaped Nd magnet($\phi 20 \times 5mm$, 12g) rolls down. Its length(l) is 1.2m. Its thickness(t) is 3mm. Its widths(ω) are 5, 6, 7, 8cm. A piece of section paper which contains $2 \times 2cm$ squares on it is attached on the copper plate to measure the magnet's displacement per time.

3-1-2. An inclined angle(θ)

This is the parameter which is to understand how the motion of a magnet will be varied as gravitational force($mg \sin \theta$) is increased. Also, the magnitude of the electromagnetic force that pushes a rolling magnet backward is able to be examined by this parameter.

3-1-3. An angle between the x-axis(the direction of magnet's movement) and the North Magnetic Pole(ϕ)

This is the parameter which is to examine how the earth's magnetic field affect the motion of a rolling magnet. Set the North Magnetic Pole, 0° , and the east, 90°

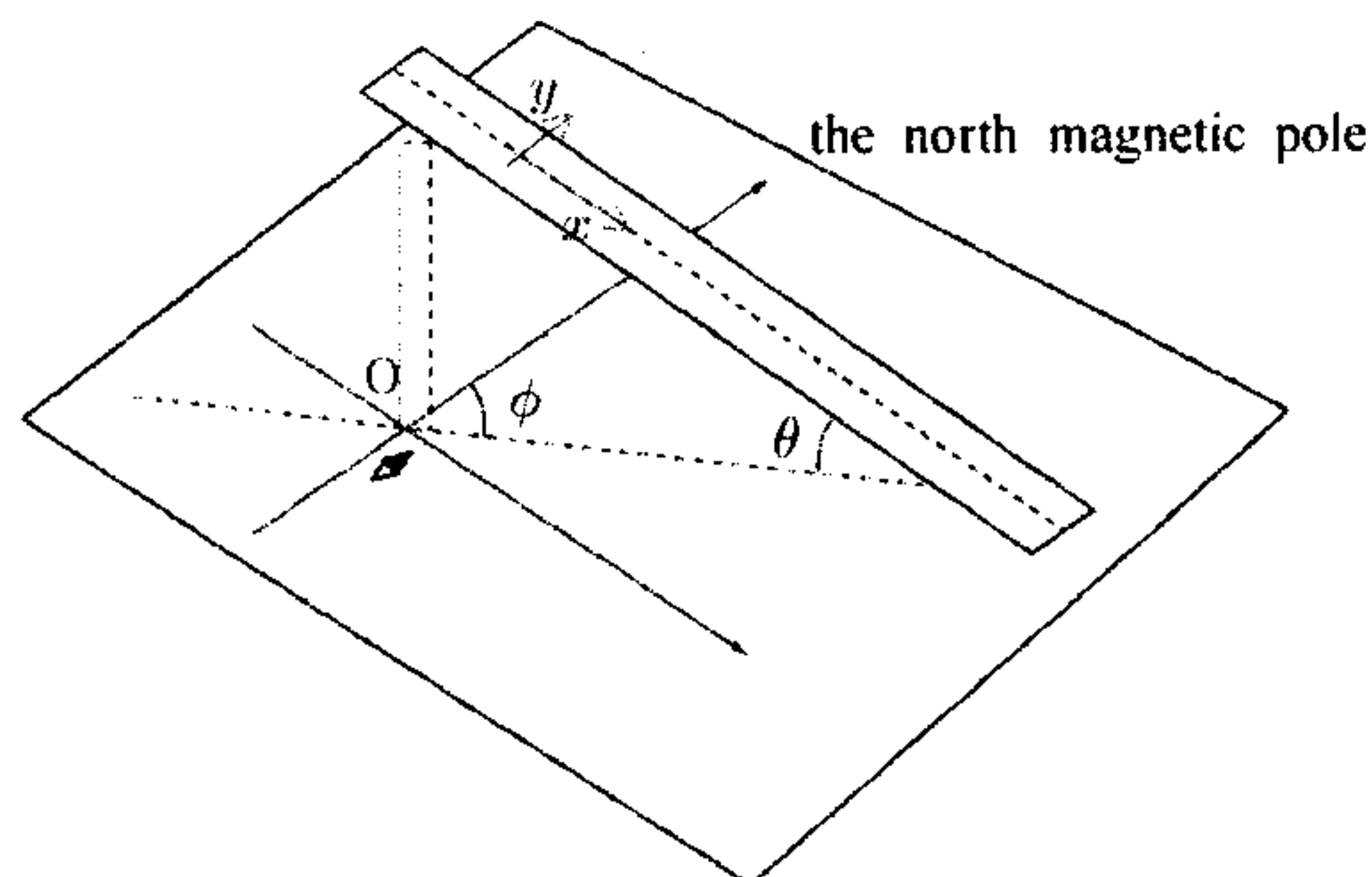


Figure 2. The adopted coordinate system

3-1-4. Interval time

Using a digital camcorder, record the entire experiments and change the video files of experiments to JPG files; the interval time between JPG files is 1/30 of a second.

3-2. Analysis of the motion of a rolling magnet

The general motion of a rolling magnet which is affected by the earth's magnetic field, the magnetic field induced by eddy currents, gravitational force and frictional force is like the below picture.



Figure 3. The motion of a magnet on a copper plate (when, $\phi = 90^\circ$, $\theta = 13^\circ$, $\omega = 6cm$, $\Delta t = 0.1sec$, the initial position on a plate $y = -2cm$)

The factors which affect the motion of a rolling magnet are gravitational force, frictional force, the force exerted between a magnet and the magnetic field and the force exerted between a magnet and the magnetic field induced by eddy currents. Various factors affect the motion complexively, but, to analyze the motion on the x-axis and the y-axis respectively, set the model of the eddy current and the motion equation of a rolling magnet.

Also, the magnet was approximated to be as big as a big magnetic dipole, and characteristics of the magnet's motion were analyzed in proportional relation because mathematical approach to eddy current and the magnetic field induced by it is too complex to solve.

To analyze the characteristic of motion on the x-axis and the y-axis respectively, the study ignored the effect of friction, and applied differently the eddy current's contribution to a rolling magnet to x-axis and y-axis.

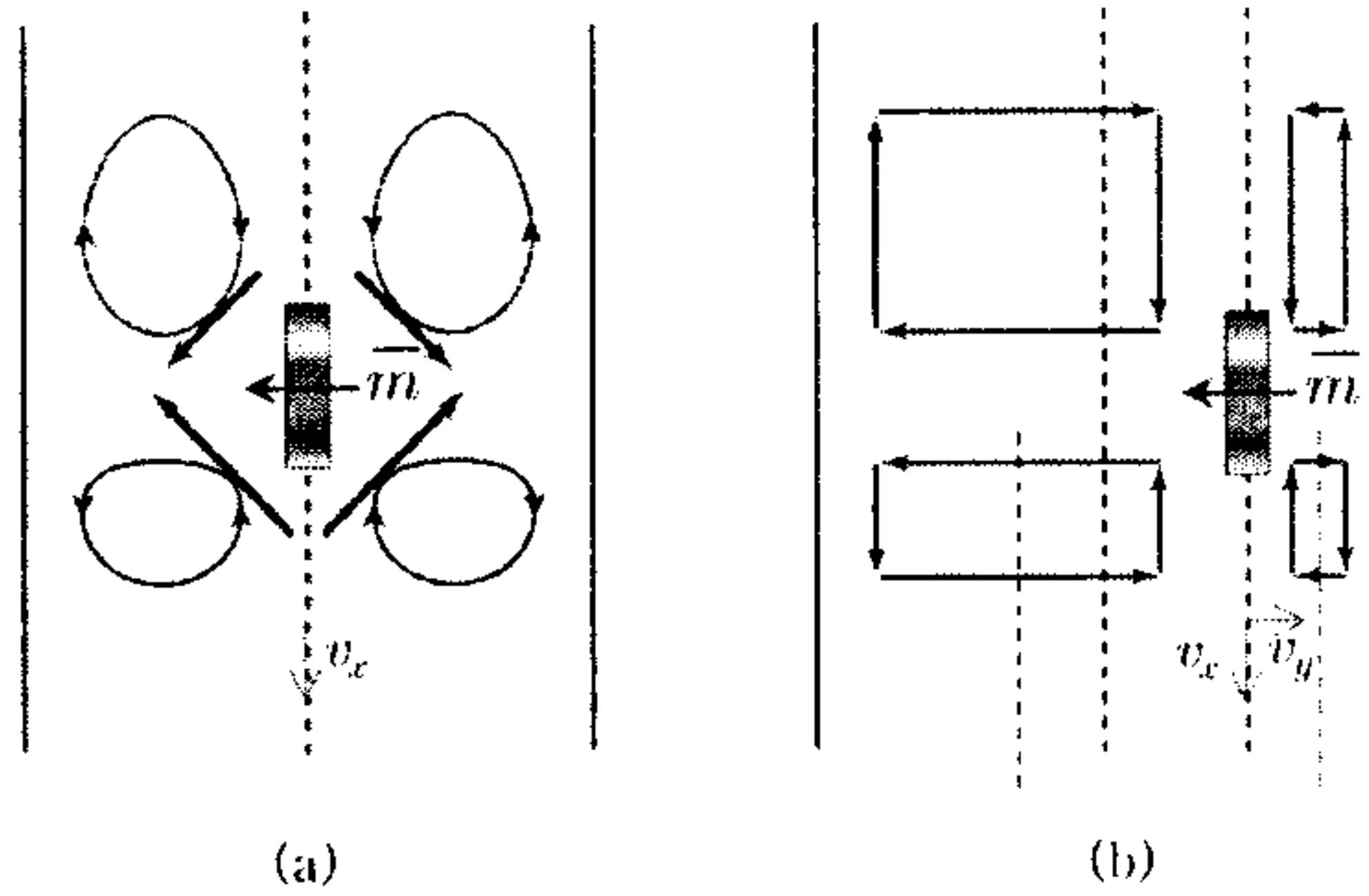


Figure 4. (a) The model of eddy current on x-axis
(b) The model of eddy current on y-axis

3-2-1. The motion of a rolling magnet on the x-axis

From $t=0$, a magnet accelerates on the x-axis, but it soon moves with uniform velocity. Therefore, it is possible to assume that gravitational force($mg\sin\theta$) is the same with the electromagnetic force generated by the eddy current.

Here, F , the electromagnetic force which exerts on $-x$ -axis is able to be approximated like below:

$$F \approx \frac{B^2 \omega^2}{4R} v \quad (23)$$

So, finally, the motion equation of a rolling magnet on x-axis is like below:

$$ma = m \frac{dv}{dt} = mg \sin \theta - c_1 v \quad (24)$$

As expressed in Figure 4(a), the Doppler Effect which occurs as a magnet rolls down is applied to express the model of eddy currents on the metallic plate, and the eddy currents generated in front of a magnet are more powerful to interrupt a rolling magnet than one generated behind a magnet. Based on Lenz's law, in both cases, electromagnetic forces exert to hinder the magnet's movement.

3-2-2. The motion of a rolling magnet on the y-axis

When a magnet starts to roll down at the initial position deviated from the center line, it oscillates because of the force of restitution which is generated by the difference of the electromagnetic forces of its right side and left side.

As expressed in Figure 4(b) which shows the model of eddy currents when a magnet deviates from the center line, the eddy current is considered as a tetragonal loop. When just considering the largest contribution of the loop component which is parallel to the x-axis and the closest to a magnet, the contribution of the magnetic field generated by the eddy current induced in front of a magnet and that of the magnetic field generated by the eddy current induced behind a magnet are different. So, based on the equation (24), a magnet oscillates to the center line.

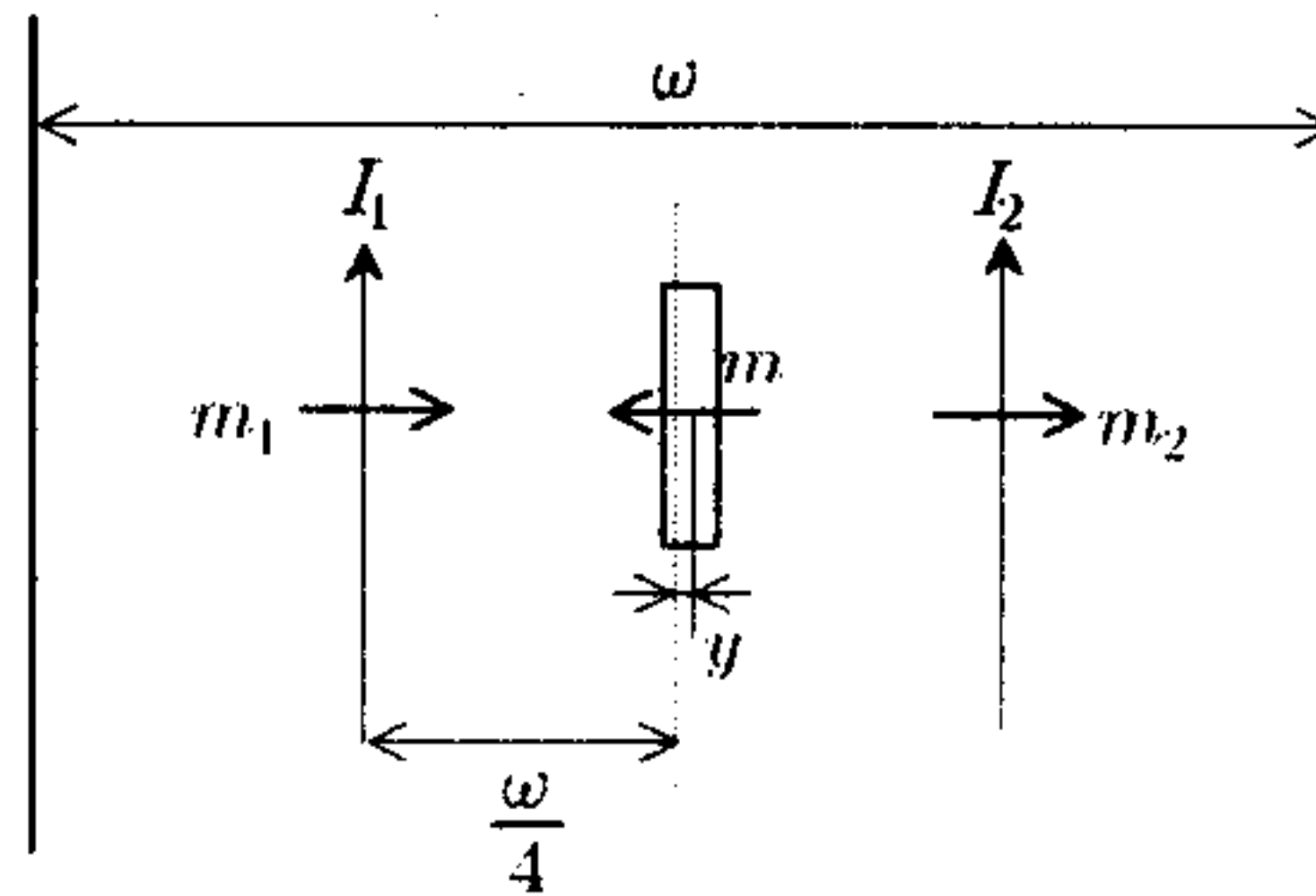


Figure 5. The diagram which is to explain the oscillation of a magnet on y-axis

The eddy current is considered as a linear current Figure 9 shows, and the position of it is approximated at the center of the area from the edge of a copper plate to that of

a magnet. At the position where a magnet is placed, the direction of magnetic moments which are generated by two linear currents are the same, but the effect on the y-axis that affects a magnet originates from the difference of two magnetic moments.

When a magnet placed at the center is shifted with infinitesimal displacement y in the direction of the y-axis, the contribution of F_2 is stronger than that of F_1 , so the net force exerts to the center.

Therefore, the magnetic field, ΔF , which determines the motion of a rolling magnet is like below:

$$F_1 = \frac{1}{4\pi\mu_0} \frac{m_1 m}{((\omega/4) + y)^2} \quad F_2 = \frac{1}{4\pi\mu_0} \frac{m_2 m}{((\omega/4) - y)^2} \quad (25)$$

$$\begin{aligned} \Delta F = F_2 - F_1 &= \frac{m}{4\pi\mu_0} \left\{ \frac{m_2}{((\omega/4) - y)^2} - \frac{m_1}{((\omega/4) + y)^2} \right\} \\ &= \frac{m}{4\pi\mu_0} \frac{16}{\omega^2} \left\{ \frac{m_2}{(1 - (4y/\omega))^2} - \frac{m_1}{(1 + (4y/\omega))^2} \right\} \quad (26) \end{aligned}$$

Adopting the formula, $(1+x)^n \approx 1+nx$ (when, $x \ll 1$), approximate the above equation.

$$\begin{aligned} \Delta F &\approx \frac{m}{4\pi\mu_0} \frac{16}{\omega^2} \left\{ \frac{m_2}{(1 - (8y/\omega))} - \frac{m_1}{(1 + (8y/\omega))} \right\} \\ &= \frac{4m}{\pi\mu_0\omega^2} \left(\frac{(m_2 - m_1) + (m_2 + m_1)(8/\omega)y}{1 - (64/\omega^2)y^2} \right) \quad (27) \end{aligned}$$

Here, for the infinitesimal displacement y , $y^2 \approx 0$ and $m_1 \approx m_2$, so it is able to be approximated conclusively like below:

$$\therefore \Delta F \approx \frac{32m(m_2 + m_1)}{\pi\mu_0\omega^3} y = ky \quad (k = \text{constant}) \quad (28)$$

Same as the analysis of the motion on the x-axis, the electromagnetic force exerts to a magnet in the opposite direction of a magnet's movement. So, the motion equation on the y-axis of a rolling magnet is like below:

$$ma = m \frac{dv}{dt} = -ky - c_2 v \quad (29)$$

IV. Results and analysis

4-1. The motion of a rolling magnet according to the change of an angle Θ

When $\phi = 90^\circ$, an angle between the North Magnetic Pole and x-axis (the direction of magnet's movement), the initial position of a magnet $y = 1\text{cm}$, and the width of a copper plate $\omega = 6\text{cm}$, an angle Θ is varied from 10° to 22° increasing 3° each interval. In these conditions, the entire motion of a rolling magnet is recorded.

Table 1. The displacement of a magnet according to the change of an inclined angle θ

$\theta = 10^\circ$			$\theta = 13^\circ$			$\theta = 16^\circ$			$\theta = 19^\circ$			$\theta = 22^\circ$		
t(sec)	x(cm)	y(cm)	t(sec)	x(cm)	y(cm)	t(sec)	x(cm)	y(cm)	t(sec)	x(cm)	y(cm)	t(sec)	x(cm)	y(cm)
0.00	0.0	-1.0	0.00	0.0	-1.0	0.00	0.0	-1.0	0.00	0.0	-1.0	0.00	0.0	-1.0
0.23	2.5	0.0	0.47	11.0	0.0	0.27	3.0	0.0	0.10	6.0	0.2	0.03	2.5	0.0
0.53	10.0	0.7	0.73	20.5	0.8	0.47	10.0	1.1	0.27	12.5	1.1	0.20	9.5	1.0
0.83	17.5	0.2	0.97	28.5	0.0	0.70	19.0	0.2	0.47	21.5	0.1	0.40	19.7	0.0
1.20	28.0	-0.7	1.20	36.5	-0.7	0.93	28.5	-1.1	0.67	31.0	-1.2	0.57	28.5	-1.0
1.60	38.0	0.0	1.50	47.0	0.0	1.07	35.0	0.0	0.90	42.0	0.0	0.80	40.5	-0.2
1.83	44.0	0.3	1.73	55.0	0.5	1.27	43.2	1.1	1.07	50.0	0.9	0.97	50.0	1.0
2.10	51.0	0.2	2.00	64.3	0.1	1.47	51.5	0.2	1.23	58.0	0.0	1.13	59.0	0.0
2.50	61.5	-0.2	2.30	74.5	-0.4	1.70	61.0	-1.0	1.43	67.5	-1.0	1.30	68.0	-1.0
2.87	70.8	0.0	2.47	81.0	0.0	1.87	68.0	0.0	1.67	79.0	0.2	1.53	81.0	0.2
3.33	83.0	0.5	2.70	89.0	0.7	2.10	77.5	1.3	1.83	87.0	0.5	1.70	90.0	1.1
3.97	99.0	0.2				2.27	84.5	0.2	2.00	95.5	0.0			

4-1-1. The analysis of the motion on the x-axis

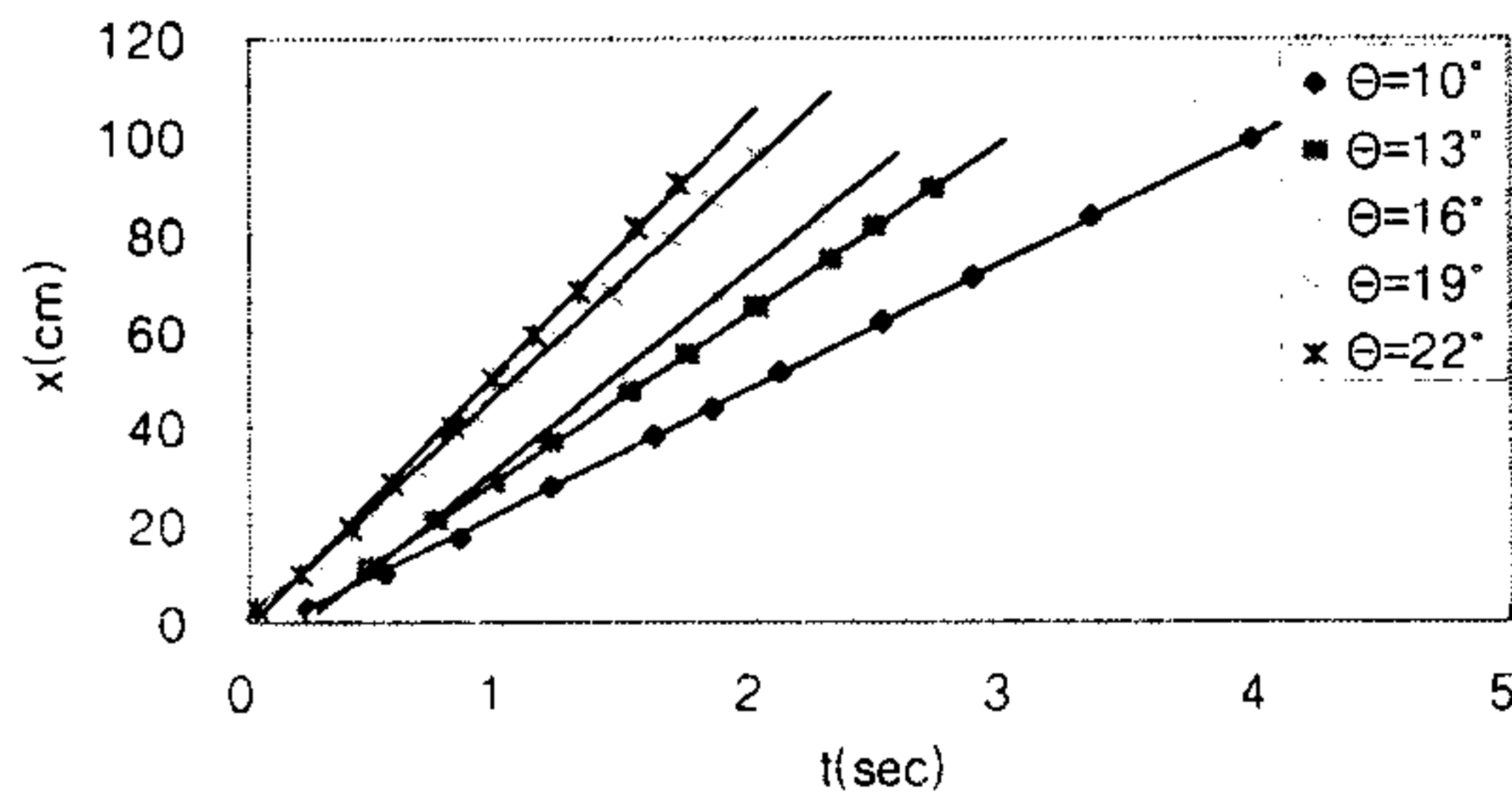


Figure 1. The displacement of a magnet on the x-axis according to the change of an inclined angle θ .

The fact that the component of gravitational force($mg\sin\theta$) and the electromagnetic force which is proportional to velocity makes equilibrium condition, a magnet roll down with uniform velocity.

$$ma = m \frac{dv}{dt} = mg\sin\theta - c_1 v \quad (30)$$

So, in this equation, $mg\sin\theta = c_1 v$.

Table 2. The velocity of a magnet according to the change of an inclined angle θ

$\theta(^{\circ})$	10	13	16	19	22
$v(\text{cm/s})$	25.96	34.83	41.23	47.42	53.11

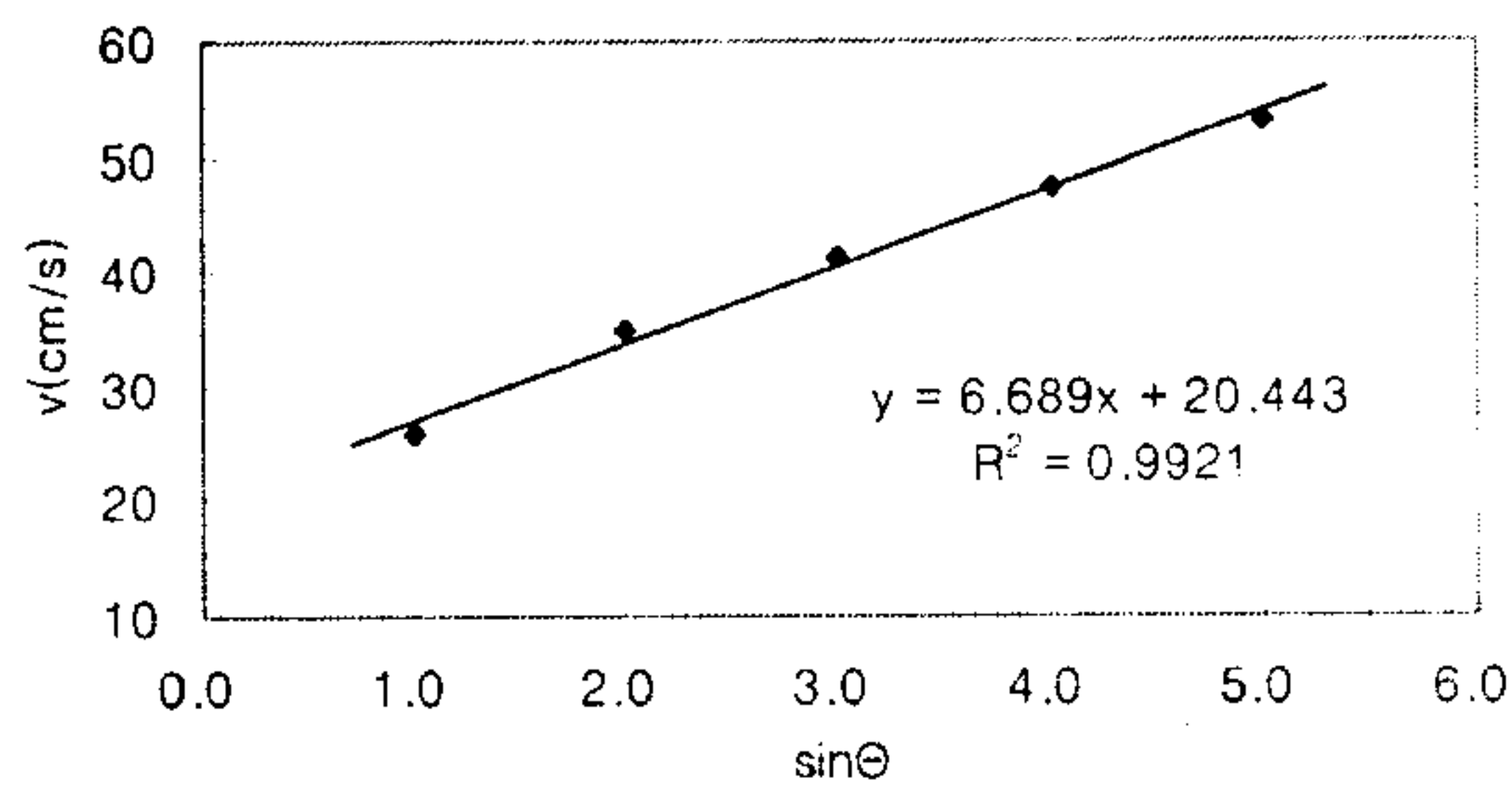
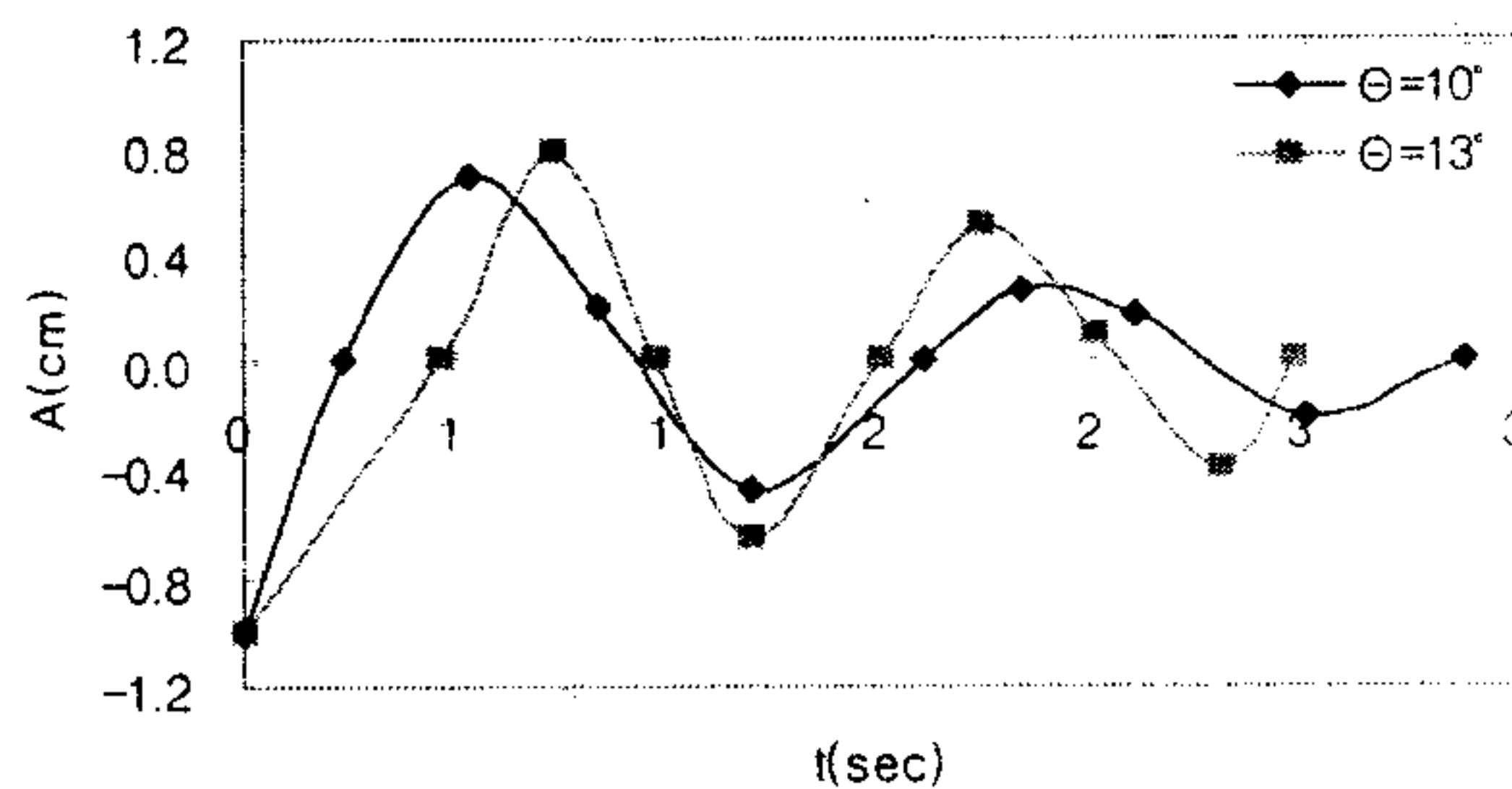


Figure 2. The velocity of a magnet according to the change of an inclined angle θ

4-1-2. The analysis of the motion on the y-axis



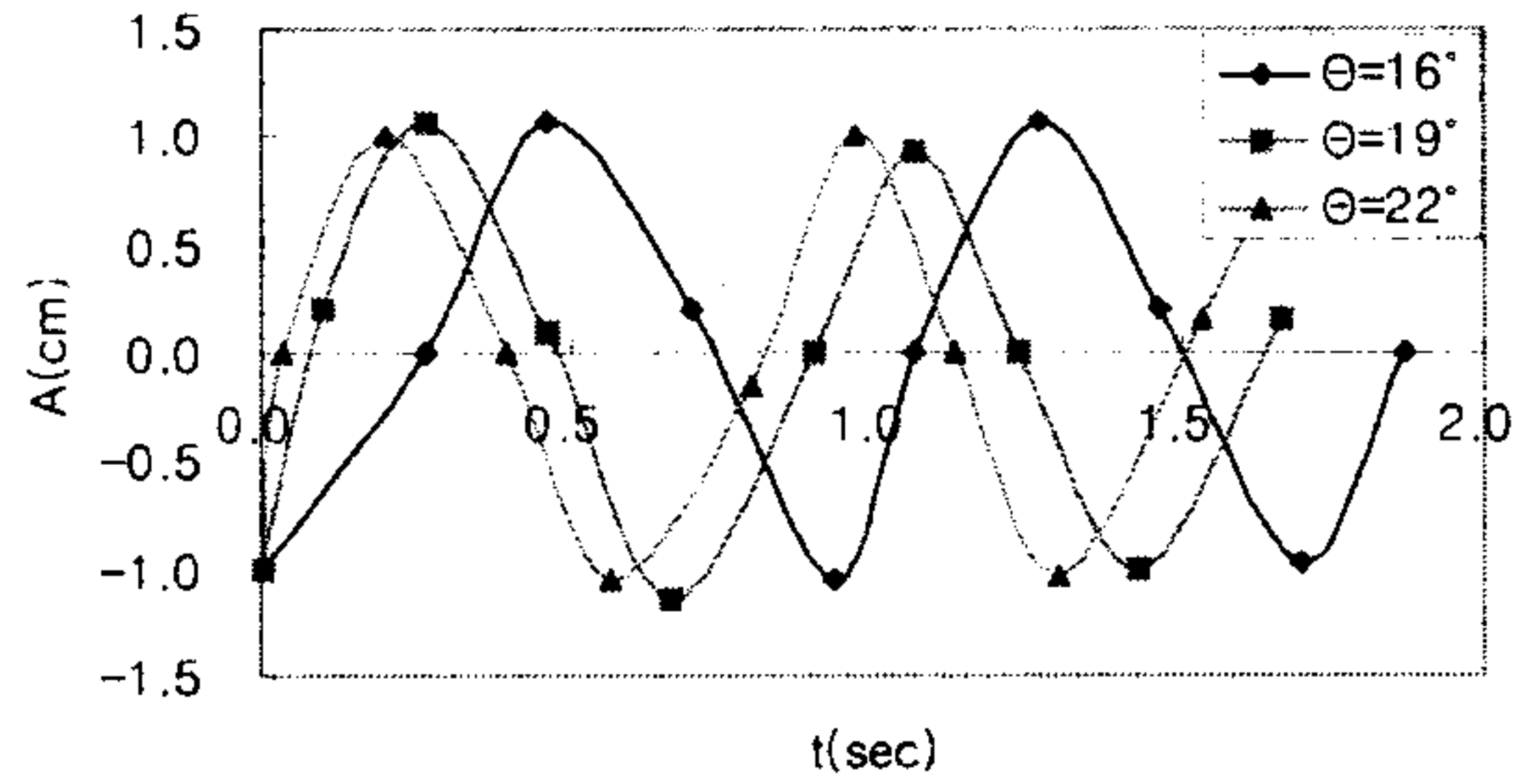


Figure 12. The graph of v_y according to the change of θ

The motion equation on the y-axis, $ma = m \frac{dv}{dt} = -ky - c_2 v$, explains the damped harmonic oscillator, and the general solution is like below:

$$y(t) = e^{-\gamma t} A \cos(\omega_d t - \psi) \quad (31)$$

And, here,

$$\gamma = \frac{c_2}{2m}, \quad \omega_d = \sqrt{\frac{k}{m} - \frac{c_2^2}{4m^2}} = \sqrt{\omega_0^2 - \gamma^2} \quad (32)$$

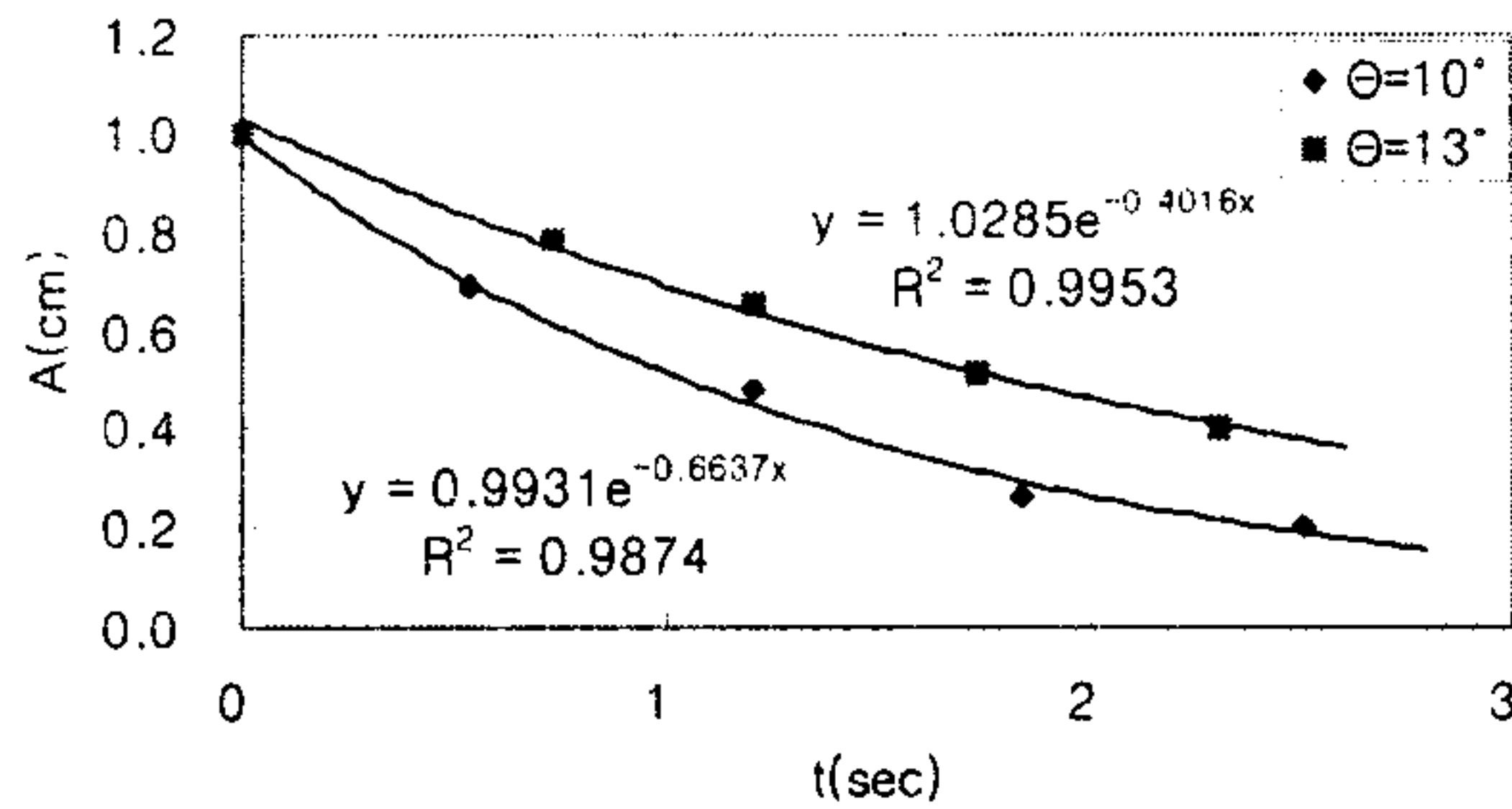


Figure 5. The change of amplitude of oscillation on the y-axis

Based on γ get from the above results, constants, k and c_2 are able to be decided. When $\theta = 10^\circ$, $\gamma = 0.66$, T (period)=1.3sec and m (the mass of a magnet)=12g, it is decided that $c_2 = 1.6 \times 10^{-2}$, $k = 2.9 \times 10^{-1}$. When $\theta = 13^\circ$, $\gamma = 0.40$, T (period)=1sec, m (the mass of a magnet)=12g, it is decided that $c_2 = 9.6 \times 10^{-3}$, $k = 4.8 \times 10^{-1}$.

The ratio(c_2/k) of two constants , k and c_2 , which are in the motion equation of a rolling magnet, $ma = -ky - c_2v$ is 5.5×10^{-2} when $\Theta = 10^\circ$ and 2.0×10^{-2} when $\Theta = 13^\circ$. As an inclined angle Θ is increased, the contribution of ky to the force exerted on the y-axis becomes bigger than that of c_2v to the force. Moreover, from when $\Theta = 16^\circ$, the contribution of c_2v is relatively small enough to be ignored, so simple harmonic motion occurs.

4-2. The motion of a rolling magnet according to the change of an angle ϕ

When an inclined angle $\Theta = 16^\circ$, the initial position of a magnet $y = 1\text{cm}$ and the width of a copper plate $\omega = 6\text{cm}$, ϕ , an angle between the North Magnetic Pole and the x-axis(the direction of magnet's movement) is varied from 0° to 90° increasing 15° . In these conditions, the entire motion of a rolling magnet is recorded.

Table 3. The displacement of a magnet according to the change of ϕ

$\phi = 90^\circ$			$\phi = 75^\circ$			$\phi = 60^\circ$			$\phi = 45^\circ$		
t(sec)	x(cm)	y(cm)	t(sec)	x(cm)	y(cm)	t(sec)	x(cm)	y(cm)	t(sec)	x(cm)	y(cm)
0.0	0.0	1.0	0.0	0.0	1.0	0.0	0.0	1.0	0.0	0.0	1.0
0.3	3.0	0.0	0.5	16.3	-0.2	0.2	8.0	1.5	0.2	6.0	1.5
0.5	10.0	-1.1	0.7	22.0	-0.4	0.4	16.0	0.2	0.5	15.5	0.0
0.7	19.0	-0.2	0.8	27.0	-0.3	0.6	21.0	-0.8	0.6	21.0	-0.6
0.9	28.5	1.1	1.1	37.0	0.5	0.8	29.0	0.3	0.7	25.0	-0.3
1.1	35.0	0.0	1.4	49.0	-0.2	1.0	38.0	1.8	1.0	35.0	1.6
1.3	43.2	-1.1	1.5	55.0	-0.3	1.2	48.0	0.2	1.2	45.0	0.2
1.5	51.5	-0.2	1.7	61.0	-0.2	1.4	55.0	-0.8	1.4	51.8	-0.9
1.7	61.0	1.0	1.9	72.0	0.6	1.6	63.0	0.5	1.5	59.0	-0.2
1.9	68.0	0.0	2.2	84.5	-0.1	1.8	72.0	1.9	1.9	65.5	1.5
2.1	77.5	-1.3	2.3	88.5	-0.2	2.0	80.5	0.4	2.0	80.0	0.2
2.3	84.5	-0.2	2.5	95.0	0.1	2.2	88.5	-0.7	2.2	86.0	-1.0
							97.0	0.0		93.0	0.0

$\phi = 30^\circ$			$\phi = 15^\circ$			$\phi = 0^\circ$		
t(sec)	x(cm)	y(cm)	t(sec)	x(cm)	y(cm)	t(sec)	x(cm)	y(cm)
0.0	0.0	1.0	0.0	0.0	1.0	0.0	0.0	1.0
0.2	5.5	2.3	0.2	5.2	2.2	0.2	5.5	2.2
0.5	17.0	0.0	0.3	9.0	2.3	0.3	9.0	2.5
0.5	19.5	-0.3	0.4	13.0	2.0	0.4	13.0	2.0
0.6	23.3	0.0	0.6	22.0	0.0	0.7	24.0	0.0
0.9	32.5	2.0	0.7	25.0	-0.3	0.8	26.5	-0.4
1.0	36.0	2.3	0.8	30.0	0.0	0.9	30.0	-0.2
1.1	41.0	1.8	1.2	44.0	2.2	1.2	45.0	2.3
1.3	51.0	0.0	1.5	57.0	0.4	1.4	53.0	1.8
1.4	54.0	-0.3	1.6	62.0	-0.3	1.6	63.0	0.4
1.6	61.0	0.0	1.7	67.0	0.2	1.7	67.0	-0.2
1.8	69.0	1.8	2.1	81.0	2.3	1.8	72.0	0.4
1.9	76.0	2.2	2.3	92.0	0.4	2.2	85.0	2.5
2.2	86.0	0.0				2.3	92.0	2.3
2.3	91.0	-0.6						

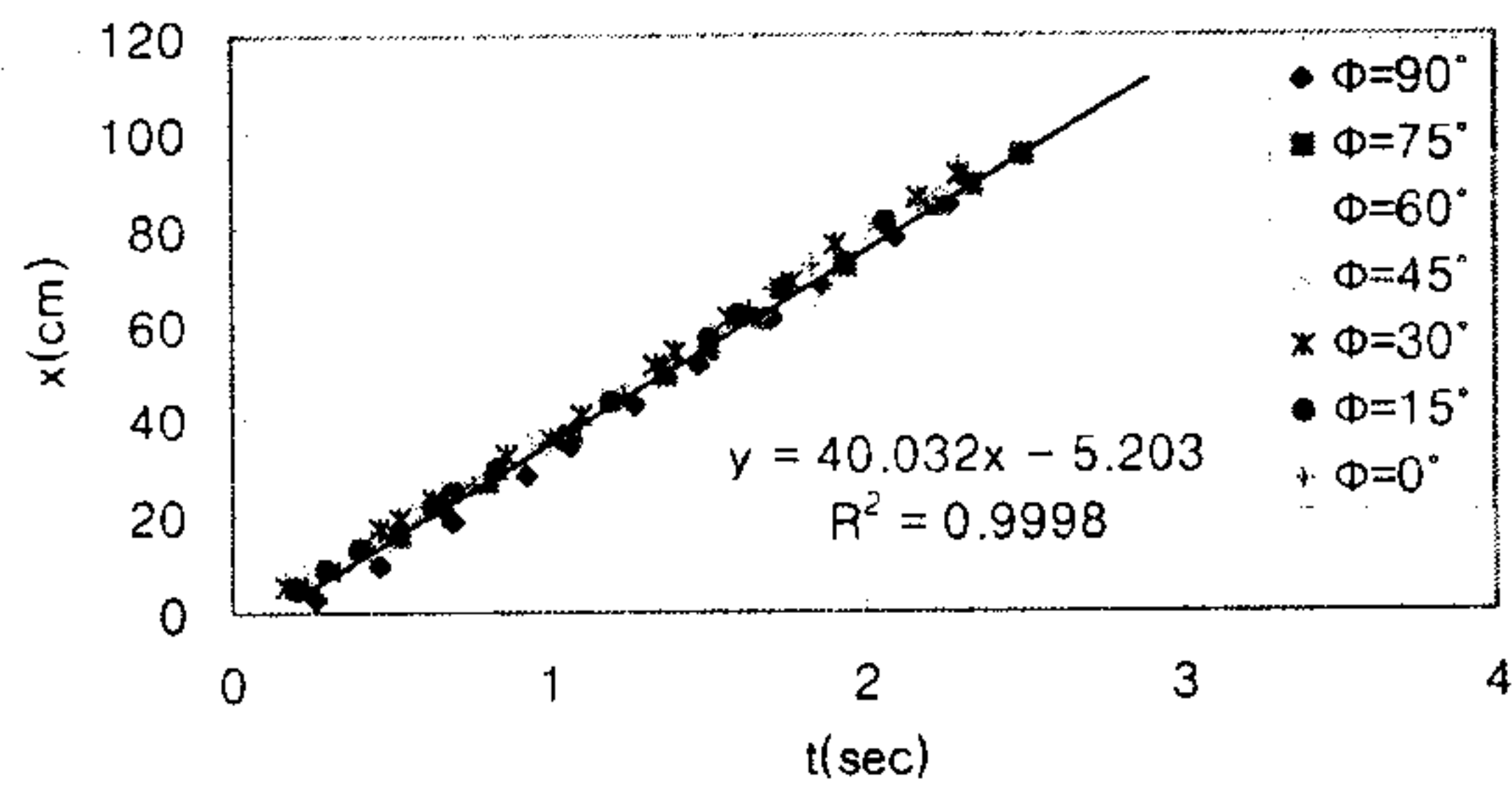


Figure 15. The graph of displacement on the x-axis according to the change of ϕ

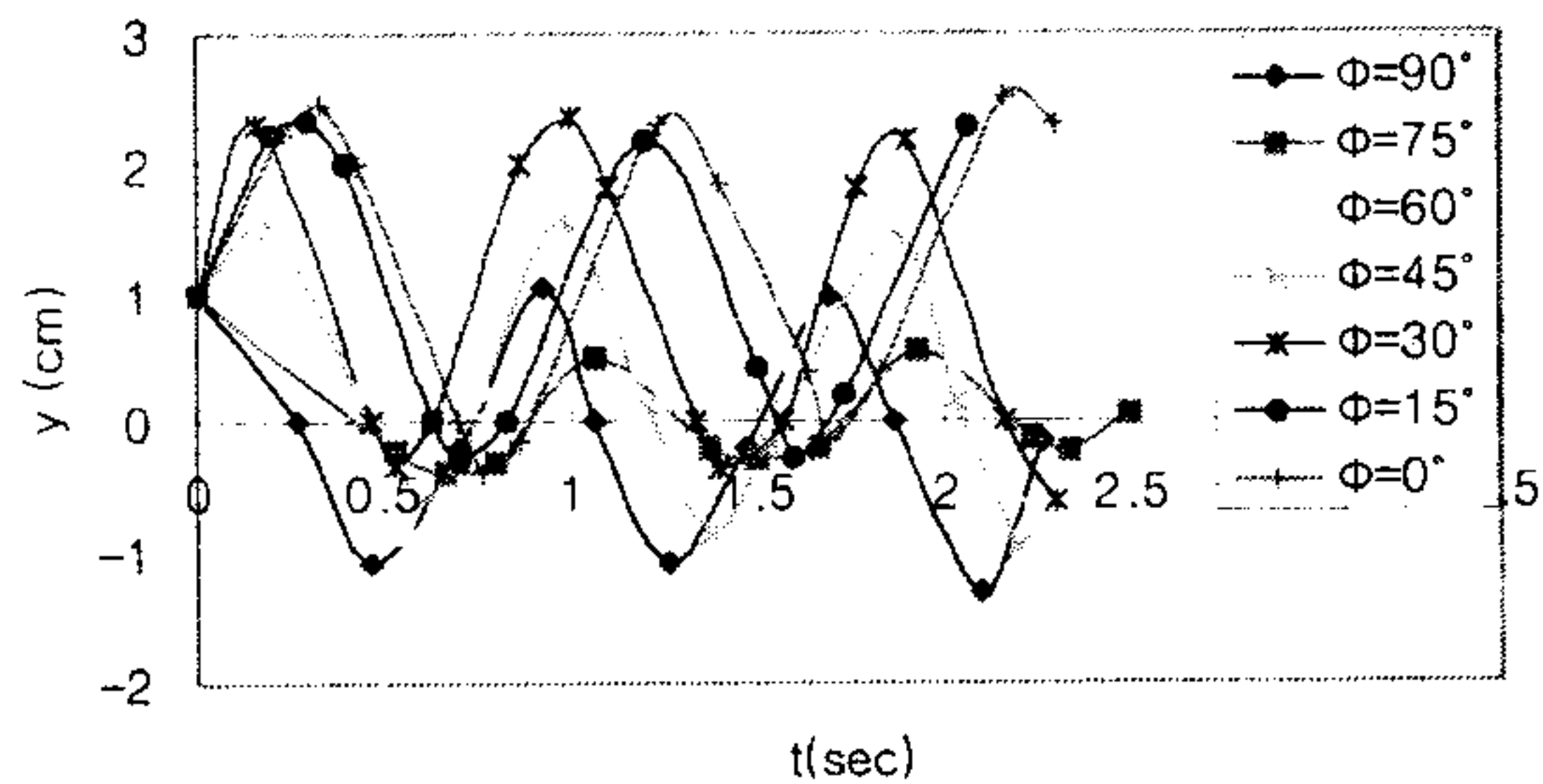


Figure 16. The graph of displacement on the y-axis according to the change of ϕ

As Figure 15. shows, the motions of a rolling magnet on the x-axis are determined by the electromagnetic force induced by a magnet and the gravitational force which depends on Θ .

However, as Figure 16. shows, the motions of a rolling magnet on the y-axis have different initial conditions as ϕ changes. When a magnet starts its motion($t=0$), the center line of oscillation and the initial phase shift due to $\tau = \vec{m} \times \vec{B}_e$, the mutual effect of a magnet(\vec{m}) and earth's magnetic field(\vec{B}_e).

The shift of the center line of oscillation is to make new equilibrium conditions. earth's is considered as a big magnetic moment(\vec{M}), and the force between these two magnetic moments(\vec{m}, \vec{M}) makes the center line of oscillation shift.

The force exerted between \vec{m} and \vec{M} is like below

$$F = \frac{1}{4\pi\mu_0} \frac{\vec{m} \cdot \vec{M}}{R^2} = \frac{1}{4\pi\mu_0} \frac{mM\cos\phi}{R^2} \quad (R: \text{radius of earth}) \quad (33)$$

Table 4. The y-coordinate of the center line of oscillation according to the change of ϕ

$\phi(^{\circ})$	90	75	60	45	30	15	0
Coordinate of the center line on y-axis(cm)	-0.03	0.10	0.49	0.77	1.23	1.25	1.35

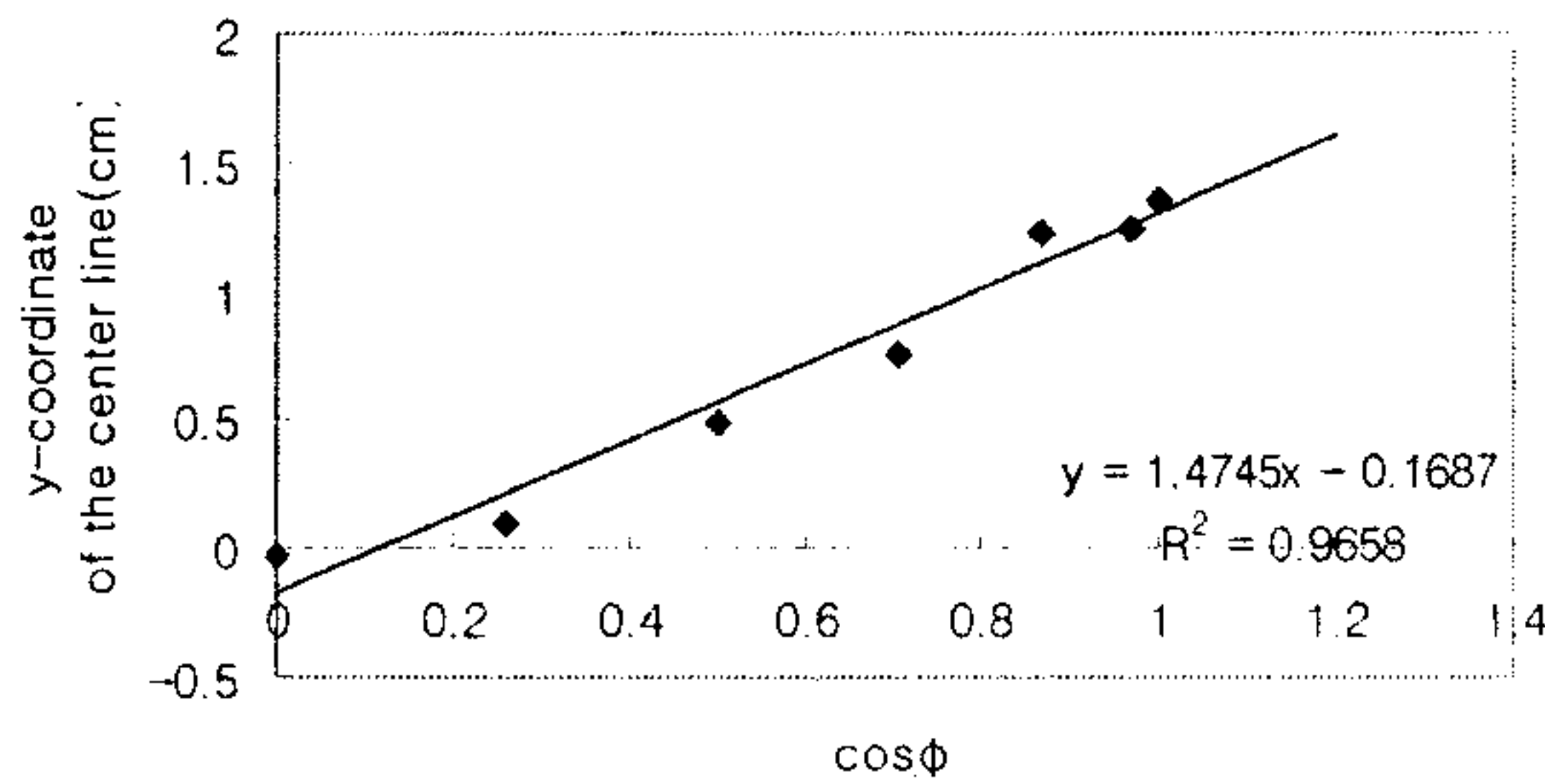


Figure 17. The graph of the center line of oscillation according to the change of ϕ

According to the change of the force ($F \propto y$) which makes a magnet oscillate, the center line of oscillation changes like Figure 17 above. The shift of the center line of oscillation is related with the force($F \propto \cos\phi$) that is, interaction between \vec{M} and \vec{m} , and the relationship between them is proportional to each other.

4-3. The motion of a rolling magnet according to the change of ω

When an inclined angle $\theta = 13^\circ$, the initial position of a magnet $y = -2\text{cm}$ and an angle between the North Magnetic Pole and x-axis (the direction of magnet's movement), $\phi = 90^\circ$, the width of a copper plate is varied from 5cm to 8cm increasing 1cm each interval. In these conditions, the entire motion of a rolling magnet is recorded.

Table 5. The displacement of a magnet according to the change of ω

$w = 5\text{cm}$			$w = 6\text{cm}$			$w = 7\text{cm}$			$w = 8\text{cm}$		
t(s)	x(cm)	y(cm)	t(s)	x(cm)	y(cm)	t(s)	x(cm)	y(cm)	t(s)	x(cm)	y(cm)
0.00	0.0	-2.0	0.00	0.0	-2.0	0.00	0.0	-2.0	0.00	0.0	-2.0
0.37	7.0	0.0	0.37	7.0	0.0	0.53	11.0	0.0	0.53	11.0	0.0
0.57	13.5	1.0	0.60	14.5	1.7	0.90	22.7	1.7	0.93	24.5	1.9
0.77	20.9	0.1	0.83	22.5	0.1	1.20	33.0	-0.1	1.53	44.0	0.0
0.93	26.5	-1.1	1.07	30.9	-1.6	1.47	41.5	-0.5	1.73	50.0	-0.2
1.17	35.0	-0.1	1.33	40.2	0.0	1.73	50.5	0.0	1.93	57.0	0.2
1.37	42.5	1.1	1.57	48.8	1.3	2.20	65.5	1.4	2.50	76.0	1.5
1.50	48.5	0.0	1.80	57.0	-0.1	2.63	80.5	-0.2	2.83	88.0	1.2
1.70	55.5	-1.1	2.00	64.0	-1.1	2.90	89.0	0.4			
1.97	65.0	-0.1	2.20	71.0	0.0						
2.17	73.0	1.1	2.47	81.0	1.1						
2.33	79.0	0.2	2.60	87.0	0.0						
2.60	89.0	-0.9									
2.87	98.0	0.0									

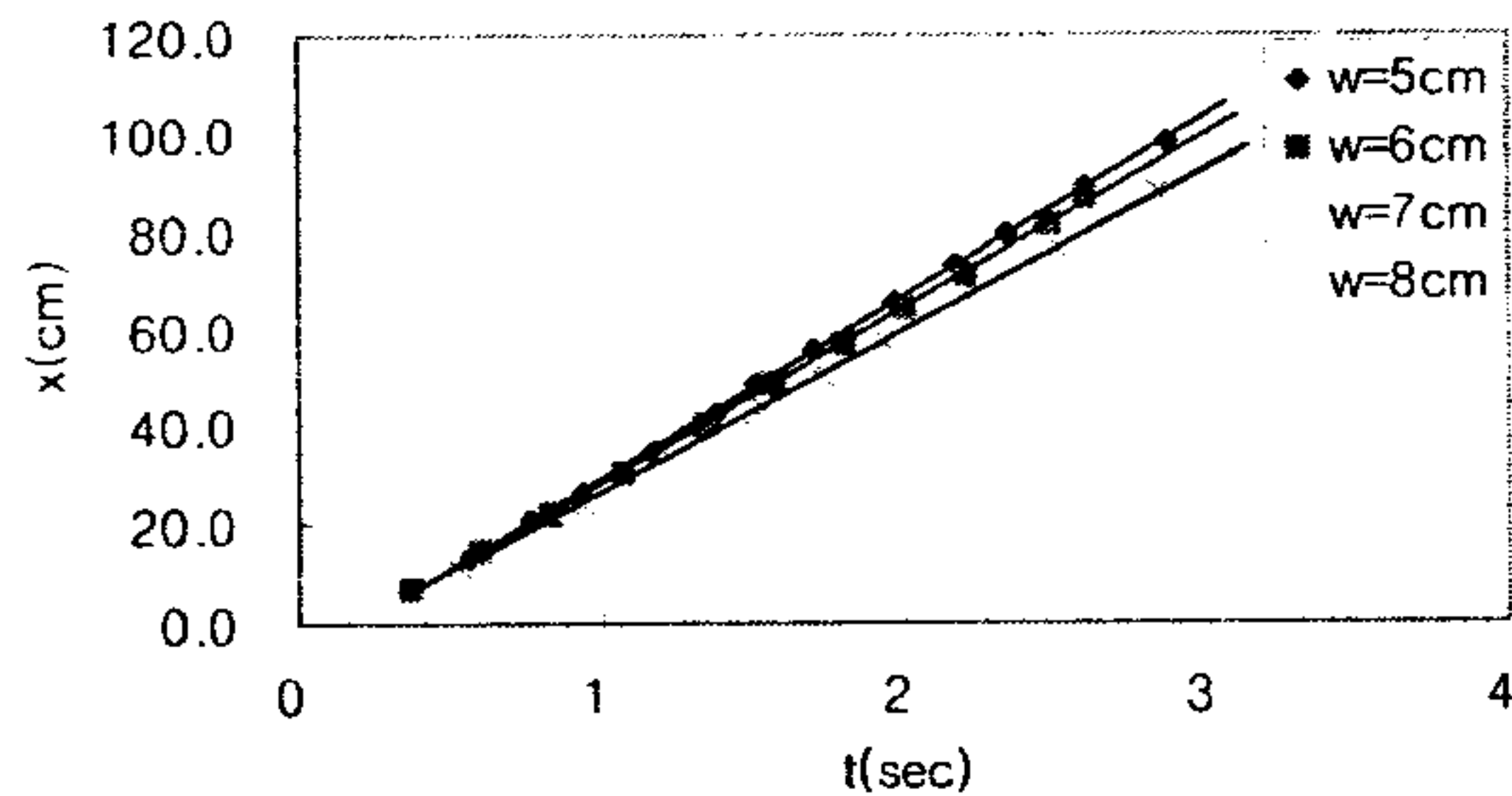


Figure 18. The graph of the displacement on the x-axis according to the change of ω

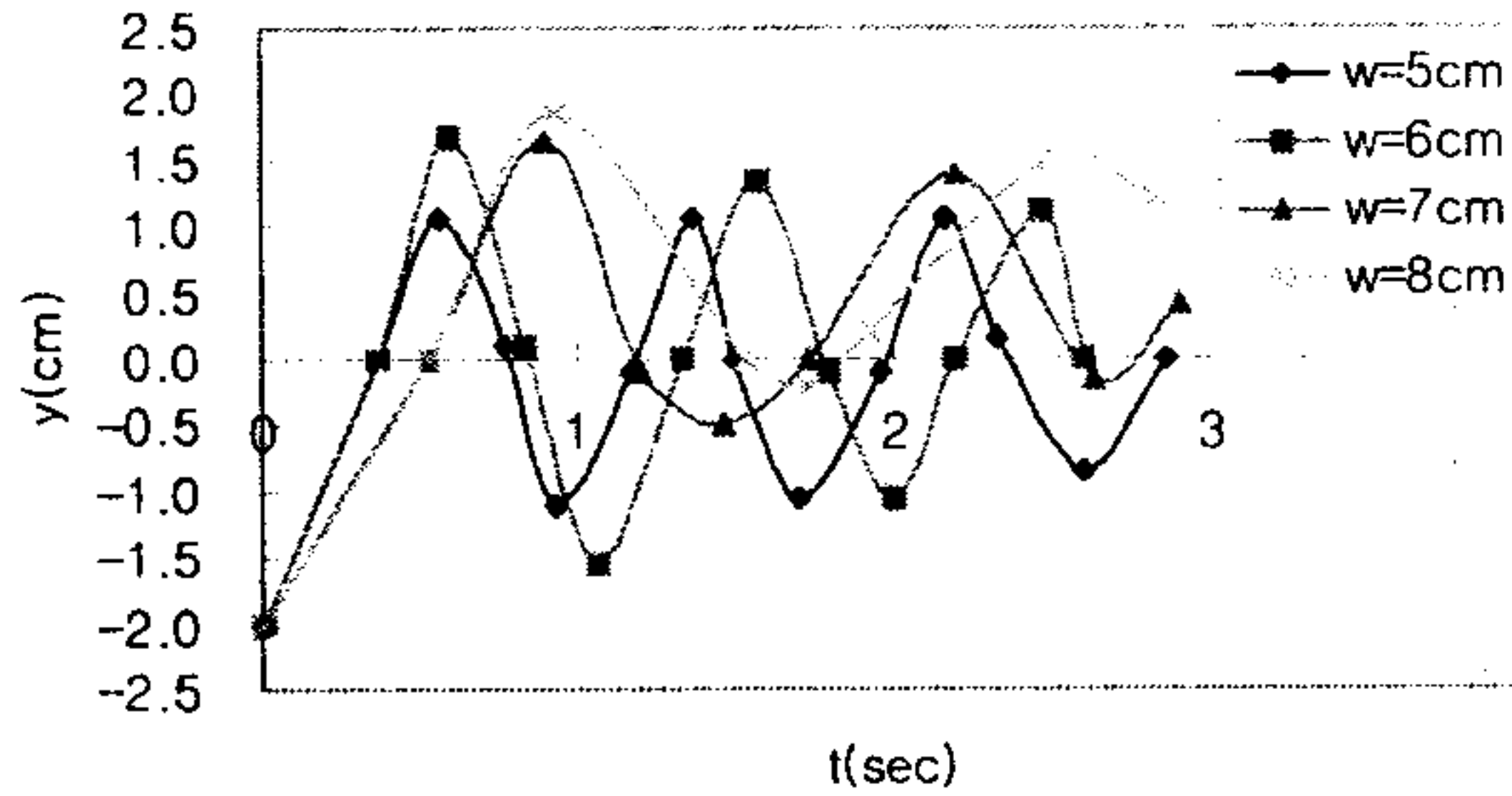


Figure 19. The graph of the displacement on the y-axis according to the change of ω

The motions of a rolling magnet on the x-axis are determined by the electromagnetic force induced by a magnet and the gravitational force which depends on Θ , but the change of the width of a copper plate hardly affect the motion of it.

Table 6. The period of oscillation according to the change of ω

$\omega (cm)$	5	6	7	8
$T(sec)$	0.80	0.97	1.30	1.57

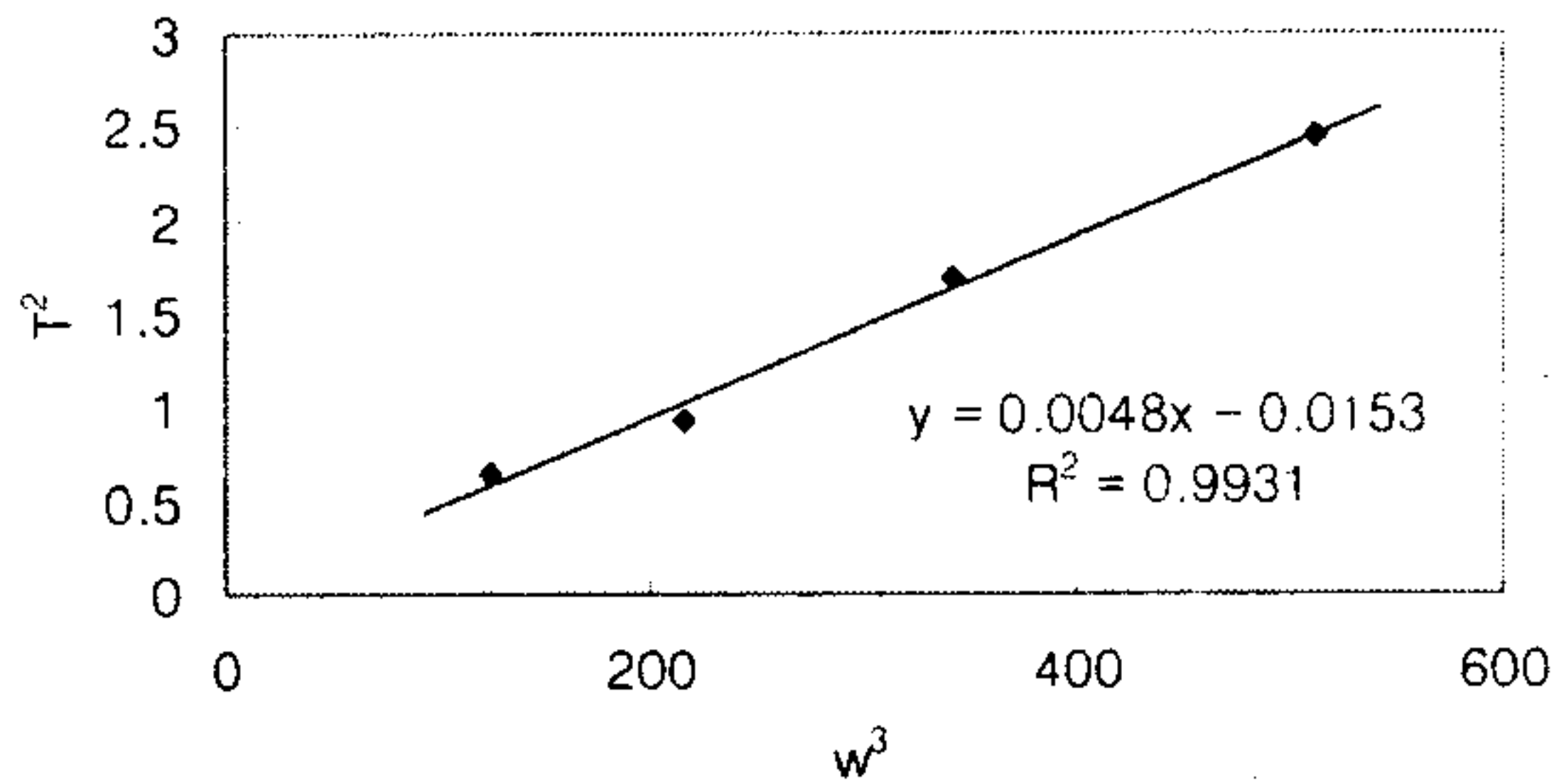


Figure 20. The graph of period of oscillation according to the change of ω

The motions on the y-axis have a tendency; the period of oscillation increases as the width of a copper plate becomes longer. From the equation below

$$\Delta F \approx \frac{32m(m_1 + m_2)}{\pi \mu_0 \omega^3} y \quad (34)$$

V. Conclusion

The study analyzed the motion of a magnet which rolled down an inclined copper plate based on the motion equation and the model of eddy currents which were able to explain the motions on the x-axis and the y-axis respectively considering the compositive effects of gravitational force, frictional force, the electromagnetic forces exerted among magnetic dipoles(between a magnet and the magnetic field generated by eddy current, and between a magnet and earth's magnetic field).

1. The effects of the earth's magnetic field and the magnetic field which was generated by eddy currents were considered as those of magnetic dipoles, and theoretical equations were set based on the interactions among these magnetic dipoles. Also, the study examined that the restitution force, F exerted on the y-axis was proportional to $\frac{y}{\omega^3}$.
2. The study analyzed the motion of a rolling magnet on the x-axis and the y-axis according to the change of an inclined angle(θ), and calculated the constants in the motion equation.
3. The study explained and analyzed quantitatively the shift of the center line of oscillation and that of the phase according to the change of an angle(ϕ) between the x-axis(the direction of magnet's movement) and the North Magnetic Pole which meant the interaction between a magnet and the earth's magnetic field.
4. The study examined that the relationship between the period(T) and the width of a copper plate(ω) is $T^2 \propto \omega^3$

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