Problem №17: Magnetohydrodynamics

13.2. Solution of Ukraine

Problem №17: Hardness

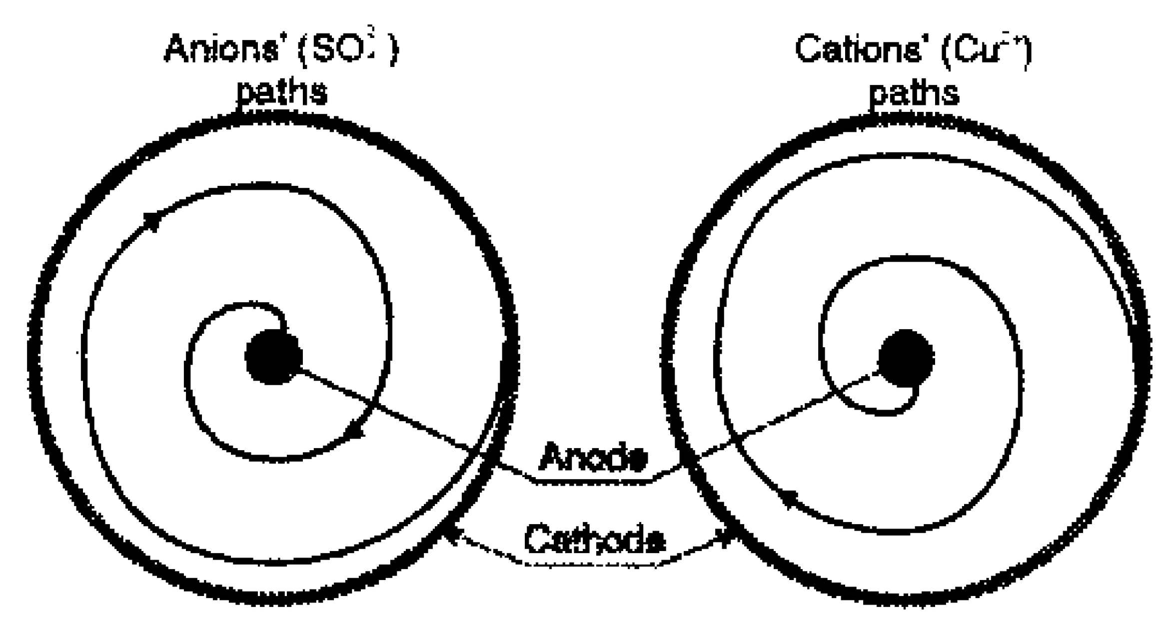
Dmytro Agarkov,
Ukrainian Physical and Mathematical Lyceum
of Kyiv National Taras Shevchenko University

The Problem

A shallow vessel contains a liquid. When an electric and magnetic fields are applied, the liquid can start moving. Investigate this phenomenon and suggest a practical application.

Introduction. Magnetohydrodynamics studies the influence of electromagnetic field on the liquids and gases with high electric conductivity. Examples of such media are plasma and molten metals.
The objective of solution is to obtain velocity distribution, depending on the radius, as a main parameter. It is calculated theoretically and compared with experimental results.
Qualitative explanation. Let us consider the phenomenon in general. The liquid is electrolyte. Our team used blue vitriol $CuSO_4 \cdot 5H_2O$ in the theoretical solution and experiments.

The molecules of $CuSO_4$ split on ions Cu^{2+} and SO_4^{2-}. After connection to the direct current source, electric field appears. Its direction is radial, from center of the plate (if anode is situated in the center of the vessel and the vessel's wall is cathode). The ions of copper and acid residue migrate to the respective contacts, magnetic field that is directed up, distorts their path.



Theory.
The objective of theoretical solution is to obtain the value of liquid tangent velocity in the certain point, angular velocity and its radial dependence.
The coordinate system is shown on right:
The motion law looks as foolows:

$$m\frac{d^2\vec{r}}{dt^2} = q\vec{E} + q[\vec{v}\times\vec{B}]$$

In axes projections:

$$\begin{cases} x: \quad m\dfrac{d^2x}{dt^2} = qE + qv_yB \\ y: \quad m\dfrac{d^2y}{dt^2} = qv_xB \end{cases}$$

Let us denominate the Larmor's precession frequency of the ions as

$$\omega_c = \frac{qB}{m}$$

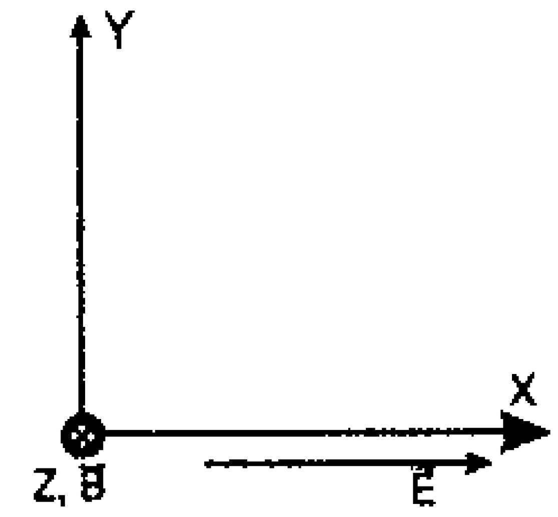
So, the system can be transformed:

$$\begin{cases} x: \quad \dfrac{d^2x}{dt^2} = \dfrac{q}{m}E + \omega_c v_y \\ y: \quad m\dfrac{d^2y}{dt^2} = -\omega_c v_x \end{cases}$$

After deriving second equation by time we obtain:

$$\frac{d^3y}{dt^3}+\omega_c^2\frac{dy}{dt}=-\omega_c\frac{qE}{m}$$

$$\frac{d^2v_y}{dt^2}+\omega_c^2v_y=-\frac{\omega_c qE}{m}$$



The general solution of the last equation is

$$v_y=C_1\cos\omega_c t+C_2\sin\omega_c t-\frac{qE}{m\omega_c}$$

$$\frac{d^3y}{dt^3}=-\omega_c\frac{d^2x}{dt^2}=-\omega_c\left(\frac{q}{m}E+\omega_c\frac{dy}{dt}\right)$$

To find constants C_1 and C_2 we use initial conditions

$$v_y(0)=v_0,\ v_y(0)=0$$

After substituting obtained values to the general solution we finally obtain:

$$v_y=-\frac{qE}{m\omega_c}(1-\cos\omega_c t)+v_0\cos\omega_c t$$

So, from the previous equation

So, we obtain the second dependence $v_y(t)$:

$$C_1=\frac{qE}{m\omega_c}+v_0$$

$$v_y=\left(\frac{qE}{m\omega_c}+v_0\right)\cos\omega_c t-\frac{qE}{m\omega_c}=-\frac{qE}{m\omega_c}(1-\cos\omega_c t)+v_0\cos\omega_c t$$

Expanding $\omega_c t$ into the Taylor's series, we obtain $\cos(\omega_c t)\approx 1-(\omega_c t)^2/2$ so

$$v_y(t)=-\frac{qE}{m\omega_c}\frac{\omega_c^2t^2}{2}+v_0-v_0\frac{\omega_c^2t^2}{2}$$

So, we can obtain velocity that ion has before collision with water molecules:

$$\Delta v_y=v_y-v_0=-\frac{\omega_c^2t^2}{2}\left(v_0+\frac{qE}{m\omega_c}\right)$$

Velocity of the ions drift is calculated as:

$$\frac{qE}{m\omega_c}=\frac{qE}{m}\frac{m}{qB}=\frac{E}{B}\sim 10^3\ m/s$$

Now we neglect with the start velocity of ions v_0 because

$$v_0\sim\frac{qE}{m\omega_c}$$

The mobility of the ions is $\mu=\frac{e}{m\text{n}}$, where **ν** is the time between two collisions.

So, finally we obtain:

$$\left|\Delta v_y\right| = \frac{\omega_c^2 t^2}{2}\frac{E}{B} = \frac{E}{2B}\left(\frac{\omega_c}{\text{n}}\right)^2$$

We suppose, that liquid moves uniformly, so $F_{Fr} = F_{im}$; here F_{Fr} is viscosity friction force, F_{im} is the force of interaction between ions and water molecules.
From the formula for viscosity friction force:

$$F_{Fr} = -\eta \frac{dv}{dz} S$$

For the linear estimation of height dependence of the velocity:

$$F_{Fr} \approx -\frac{\eta S v_{\max}}{d},$$

where d is a vessel height, S – its square, v_{max} – value of liquid velocity at the surface.

Dividing expression of the viscosity friction by the volume we obtain

$$\frac{dF_{Fr}}{dV} = -\frac{\eta v_{\max}}{d^2}$$

So one can get the expression for the force of interaction between ions and water molecules:

$$F_{im} = \rho_i \Delta v_y \nu,$$

where ρ_i is mass of ions in the unit volume of the solution.

So, equality of forces for the water is:

$$\eta \frac{v_{\max}}{d^2} = \rho_i \Delta v_y \text{n}$$

The value of ν can be found from the definition of the ionic mobility:

Substituting this value to the equality of forces we obtain:

$$\text{n} = \frac{e}{m\mu} \qquad v_{\max} = \frac{\rho_i E d^2}{2B\eta}\left(\frac{\omega_c}{\text{n}}\right)^2 \text{n}$$

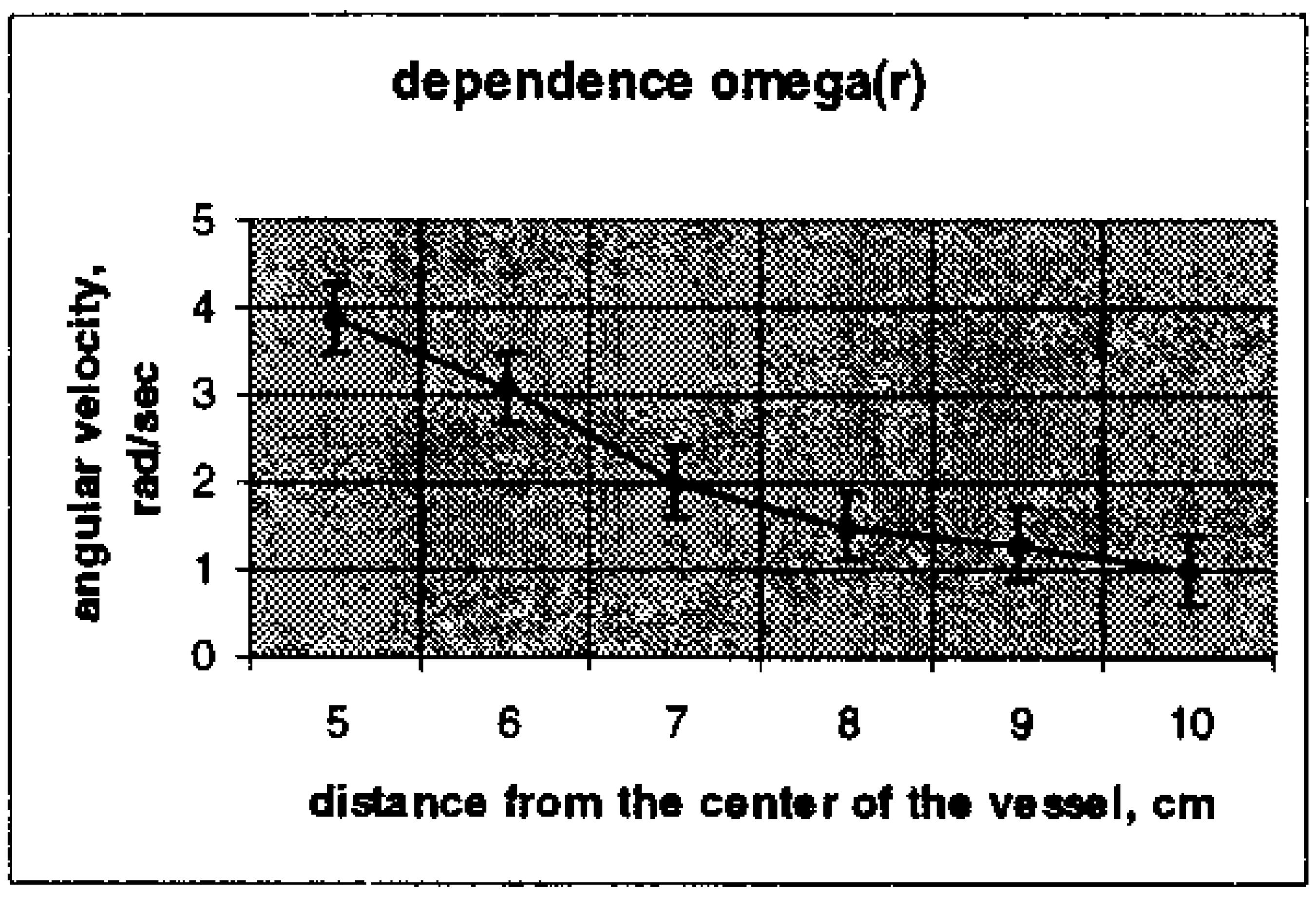
So, final formula for liquid velocity is:

$$v_{\max} = \frac{2\rho_i e E B \mu d^2}{m\eta}$$

Using the dependence of intensity on the distance from the center of the vessel r $(E\sim 1/r)$ and well-known formula of angular velocity on this distance we can obtain:

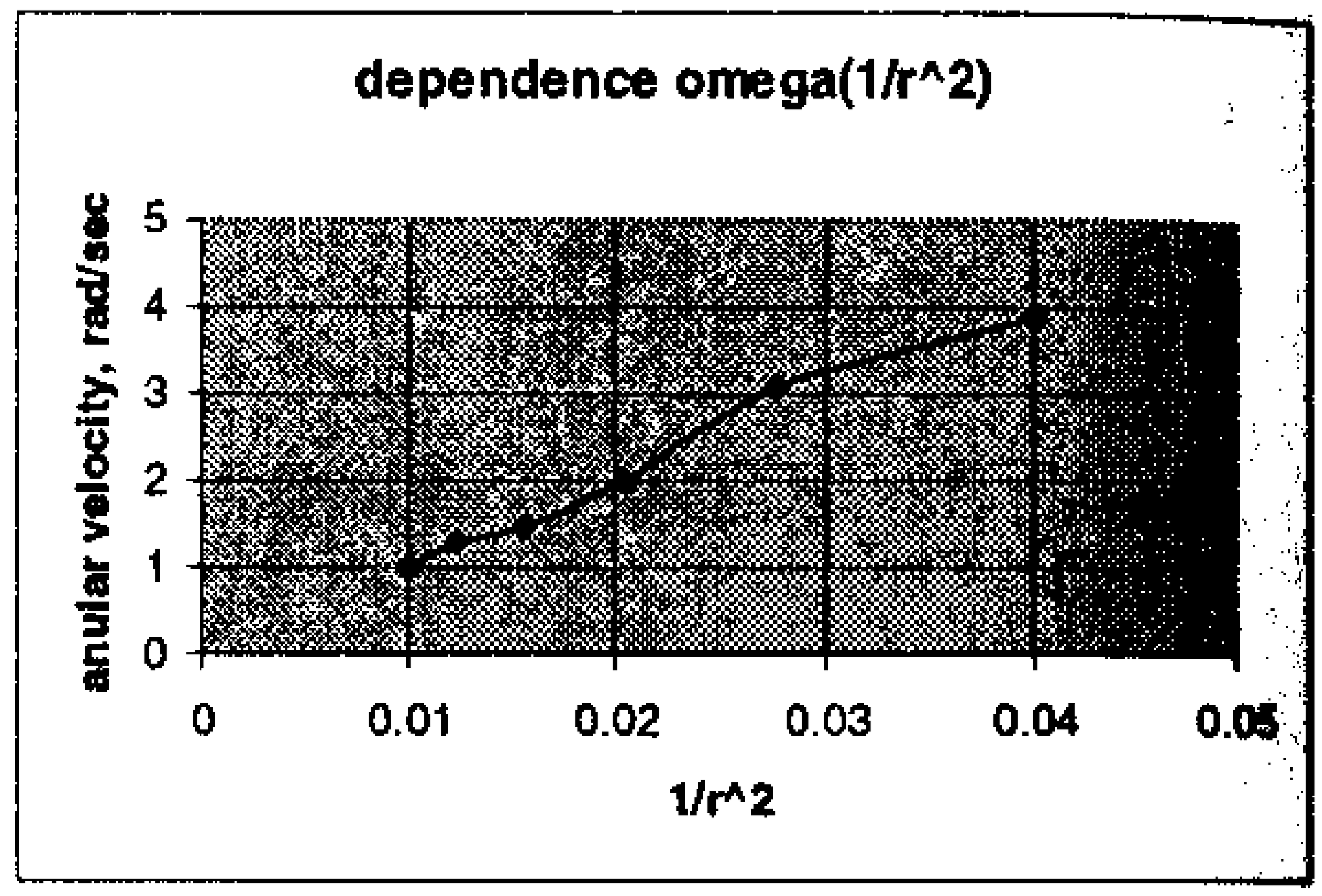
$$v_{\max} \sim E \sim \frac{1}{r}$$

$$\omega = \frac{v_{\max}}{r} \sim \frac{1}{r^2}$$



Dependence of angular velocity on the distance from the center of the vessel

Linearization of the dependence



Substituting following values to the final formula, one can obtain the numerical value for liquid velocity: $v_{max}=8.5\ cm/s$.

To investigate the effect practically, our team assembled an experimental system. Plastic plate was filled with saturated water solution of $CuSO_4$, its concentration was *19% (15°C)*.Two electric contacts were connected to the center of the plate and to its walls. They were also connected to the source of direct current *(30V)*. The rotation of the liquid was observed.

The maximal measured velocity was 3.7 cm/s, so the order of its theoretical value was determined correctly. The difference between both values can be explained with neglecting with viscosity friction of the walls, non-uniformity of the field and linearization of the *v(z)* dependence.